Representations of quantum toroidal algebras

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Quantum toroidal algebras

Construction

- Affinizations of Lie algebras:

$$\mathfrak{g} \xrightarrow{\text{affinization}} \mathbb{C}[t,t^{-1}]\otimes\mathfrak{g} \xrightarrow{\text{affinization}} \mathbb{C}[s^{\pm 1},t^{\pm 1}]\otimes\mathfrak{g}$$

Simple Lie algebra — Loop algebra — Double loop algebra

- Affinizations of quantum groups:

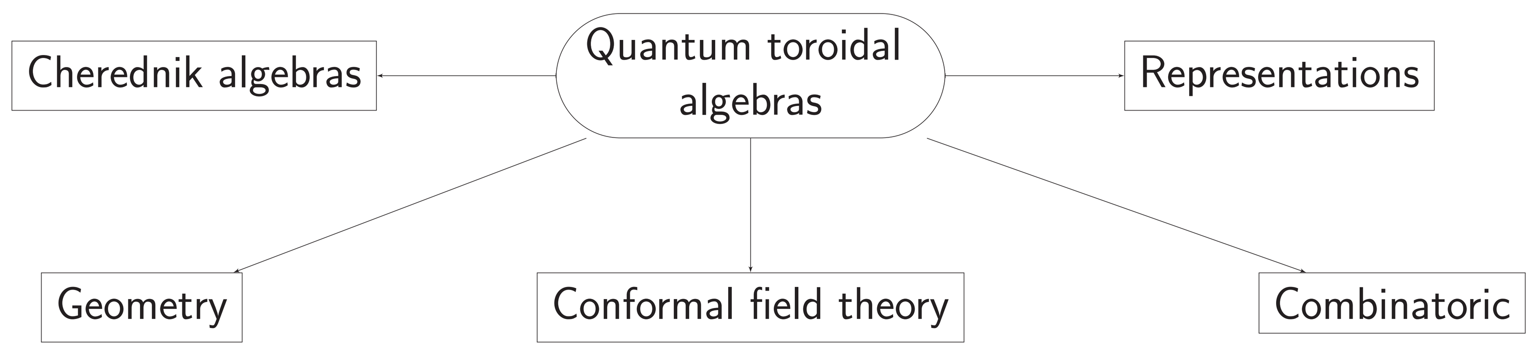
$$\mathcal{U}_q(\mathfrak{g}) \xrightarrow{\text{affinization}} \mathcal{U}_q(\hat{\mathfrak{g}}) \xrightarrow{\text{affinization}} \mathcal{U}_q(\mathfrak{g}^{tor})$$

Quantum group — Quantum affine algebra — Quantum toroidal algebra

Properties Some fundamental elements about quantum toroidal algebras:

- in type A, they are in Schur-Weyl duality with elliptic Cherednik algebras,
- the quantum affine algebra is a subalgebra of the quantum toroidal algebra,
- quantum toroidal algebras have a "coproduct" which involves infinite sums (Drinfeld coproduct).

Motivations



State of art no example of finite-dimensional representations were known until very recently.

Aim of my works

Aim Construct finite-dimensional representations of quantum toroidal algebras of type A at roots of unity. We have three different constructions:

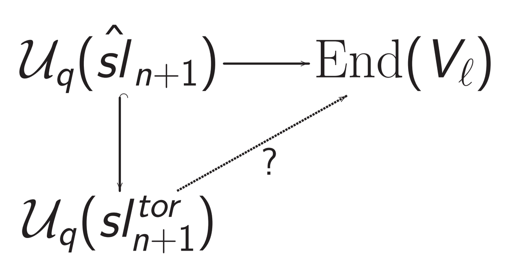
- construction via monomial crystals,
- construction by fusion products,
- construction via the affinization algebra $\mathcal{U}_q(\hat{sl}_\infty)$ of type A_∞.

Extremal representations of Kashiwara

Facts The extremal fundamental representations:

- are representations V_ℓ of $\mathcal{U}_q(\hat{sl}_{n+1})$ ($\ell = 1,\dots,n$) with crystal bases $\mathcal{B}_\ell$,
- are isomorphic to the global Weyl modules [Chari, Pressley 00],
- admit an irreducible quotient of finite dimension [Kashiwara 02].

Idea Extend the action of the quantum affine algebra on V_ℓ to an action of the quantum toroidal algebra: the representations of $\mathcal{U}_q(sl_{n+1}^{tor})$ hence obtained should have finite-dimensional quotients.



First construction

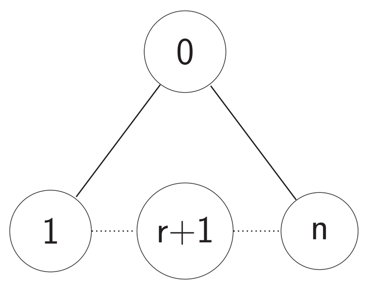
Facts

- Crystal bases $\mathcal{B}_\ell$ can be realized by monomial crystals $\mathcal{M}_\ell$ [Hernandez, Nakajima 06].
- Monomials occuring in these crystals appear also in the theory of q-characters of quantum toroidal algebras [Frenkel, Reshetikhin 99].

Aim Construct a representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ satisfying the following properties:

- its q-character is the sum of monomials in $\mathcal{M}_\ell$,
- its restriction to the quantum affine subalgebra is V_ℓ.

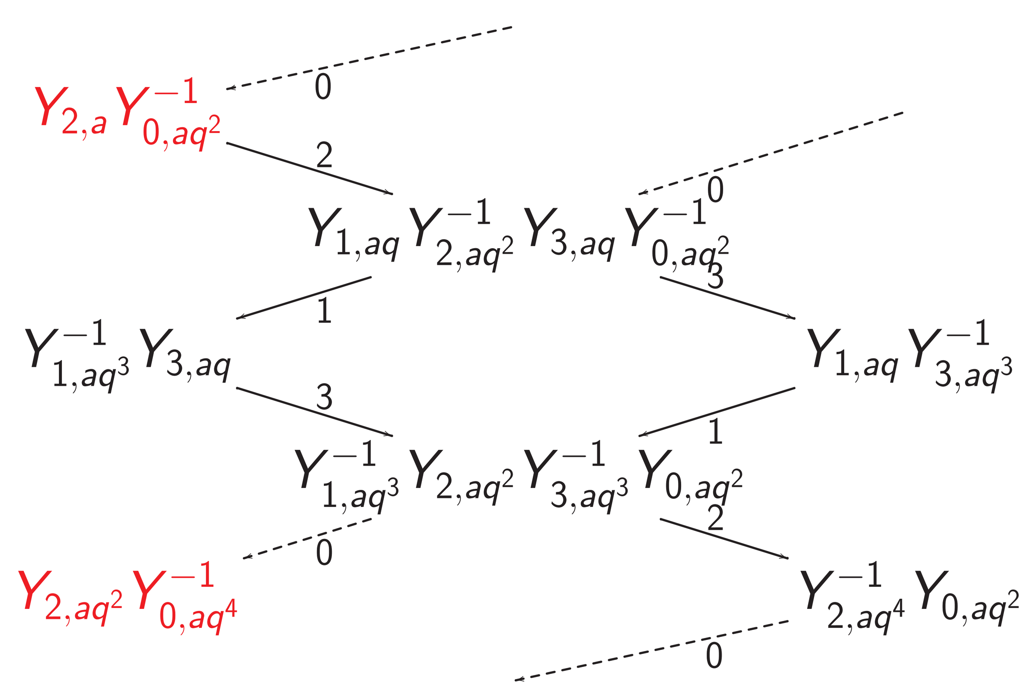
Theorem [M 14] Such a representation exists if and only if ℓ is one of the nodes $1, r+1, n$ of the Dynkin diagram, where $n = 2r+1$ is odd. It is denoted by $V_\ell(a)$ with $a \in \mathbb{C}^*$ and is called extremal loop weight representation.



Remark The extremal loop weight representations $V_1(a)$, also called vector representations, are used in [Feigin, Jimbo, Miwa, Mukhin 13].

Finite-dimensional representations

Remark The monomial crystals $\mathcal{M}_\ell$ admit shift automorphisms. This is related to the existence of finite-dimensional representations of quantum toroidal algebras at roots of unity.



Crystal $\mathcal{M}_2$ for $\mathcal{U}_q(\hat{sl}_4)$

Theorem [M 14] Specializing q at a particular root of unity in the representations $V_\ell(a)$, we get irreducible finite-dimensional representations by taking a quotient.

Remark This is the first systematic construction of finite-dimensional representations of quantum toroidal algebras at roots of unity.

Second construction

Motivation The extremal representations are related to tensor products of highest weight representations and lowest weight representations [Kashiwara 94].

Theorem [M 13]

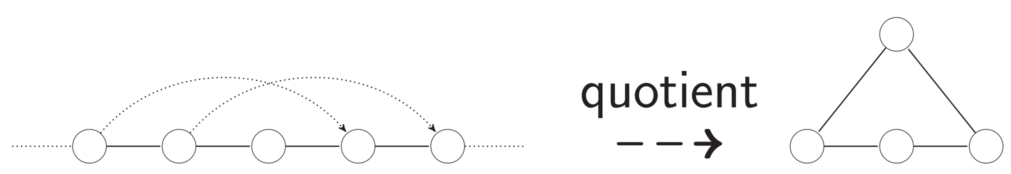
- Process of tensor products of ℓ-highest weight representations and ℓ-lowest weight representations of $\mathcal{U}_q(sl_{n+1}^{tor})$.
- We recover the vector representation $V_1(a)$.

Proof Drinfeld coproduct and related methods [Hernandez 07].

Theorem [M 13]

- We get extremal loop weight representations as subquotients of $\bigotimes_i V_1(a_i)$.
- We obtain new finite-dimensional representations at roots of unity.

Third construction



Conjecture [Hernandez 11] Relation between the q-character of representations of $\mathcal{U}_q(\hat{sl}_\infty)$ and the one of representations of $\mathcal{U}_q(sl_{n+1}^{tor})$.

Theorem [M 13-1]

- Construction of ℓ-extremal representations $V_\ell^\infty(a)$ for $\mathcal{U}_q(\hat{sl}_\infty)$.
- Proof of the conjecture: we recover the representations $V_\ell(a)$ of $\mathcal{U}_q(sl_{n+1}^{tor})$.

Perspectives

- Construction of finite-dimensional representations for quantum toroidal algebras of general type.
- Classification of irreducible finite-dimensional representations of quantum toroidal algebras at roots of unity.
- Description of finite-dimensional representations of elliptic Cherednik algebras at roots of unity by Schur-Weyl duality.

References

[M 14] Mathieu Mansuy. *Quantum extremal loop weight modules and monomial crystals*, Pacific Journal of Mathematics, Vol. 267 (2014), No. 1, 185-241.

[M 13] Mathieu Mansuy. *Extremal loop weight modules and tensor products for quantum toroidal algebras*, Preprint arXiv:1305.3481, 2013.

[M 13-1] Mathieu Mansuy. *Quantum extremal loop weight modules for $\mathcal{U}_q(\hat{sl}_\infty)$*, Preprint arXiv:13094298, 2013.

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