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Représentations ℓ-extrémales des algèbres toroïdales quantiques

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Extremal loop weight representations of quantum toroidal algebras

dirigée par David Hernandez

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Résumé

Représentations ℓ-extrémales des algèbres toroïdales quantiques

Résumé

Cette thèse est consacrée à l'étude des algèbres toroïdales quantiques et de leurs représentations. Dans l'esprit des travaux de M. Kashiwara et suivant la définition proposée par D. Hernandez, nous présentons différentes constructions de représentations intégrables appelées représentations ℓ-extrémales. Par spécialisation du paramètre quantique, nous obtenons des représentions de dimension finie des algèbres toroïdales quantiques aux racines de l'unité.

La première construction utilise les cristaux monomiaux. Nous interprétons les monômes apparaissant dans les réalisations des cristaux extrémaux en terme de q-caractères de représentation sur l'algèbre toroïdale quantique (via des opérateurs de promotion).

Dans la seconde construction, les représentations ℓ-extrémales sont obtenues par produit de fusion de représentations de plus haut ℓ-poids et de plus bas ℓ-poids (l'action étant définie via le coproduit de Drinfeld).

Nous proposons également une construction via l'algèbre affinisée $\mathcal{U}_q(\hat{sl}_\infty)$ et son lien combinatoire avec les algèbres toroïdales quantiques.

Mots-clefs

Groupes quantiques, théorie des représentations, algèbres affines quantiques, algèbres toroïdales quantiques, représentations extrémales, représentations ℓ-extrémales, cristaux monomiaux, opérateurs de promotion, coproduit de Drinfeld.

Extremal loop weight representations of quantum toroidal algebras

Abstract

This thesis is devoted to the study of quantum toroidal algebras and their representations. In the spirit of M. Kashiwara and following the definition suggested by D. Hernandez, we give different constructions of integrable representations called extremal loop weight representations. By specializing the quantum parameter, we get finite-dimensional representations of quantum toroidal algebras at roots of unity.

The first construction uses the monomial crystals. We read into the monomials occurring in the monomial realizations of extremal crystals in term of q-characters of representations over the quantum toroidal algebra (via promotion operators).

In the second construction, the extremal loop weight representations are obtained by fusion product of ℓ-highest weight representations and ℓ-lowest weight representations (the action being defined via the Drinfeld coproduct).

We also give a construction via the affinization algebra $\mathcal{U}_q(\hat{sl}_\infty)$ and its combinatorial link with the quantum toroidal algebras.

Keywords

Quantum groups, representation theory, quantum affine algebras, quantum toroidal algebras, extremal representations, extremal loop weight representations, monomial crystals, promotion operators, Drinfeld coproduct.

Contents

Introduction

Les groupes quantiques

Un peu d'histoire

Les *groupes quantiques* sont apparus pour la première fois en physique statistique, dans les travaux de L. D. Faddeev et de l'École de Leningrad sur les procédés de diffusion inverse (inverse scattering methods) pour résoudre des modèles intégrables. Ce sont des algèbres associatives définies à partir d'une matrice de constantes (dépendant du système intégrable considéré) appelée la R-matrice quantique. V. G. Drinfeld et M. Jimbo [Dri85, Jim85] ont montré en 1985 que ce sont des algèbres de Hopf qui, pour la plupart, sont des déformations à un paramètre de l'algèbre enveloppante universelle associée à une algèbre de Lie. Peu de temps après, S. L. Woronowicz [Wor89] et Yu. I. Manin [Man88] construisirent de façon indépendante des algèbres de coordonnées quantiques, déformations d'algèbres de fonctions sur des groupes de matrices.

La terminologie de groupe quantique s'est popularisée après son emploi par V. G. Drinfeld lors du Congrès International des Mathématiciens de Berkeley en 1986 [Dri87]. Il n'y en a pas de définition générale satisfaisante, mais il est commun d'appeler groupe quantique une algèbre de Hopf non-commutative, non-cocommutative satisfaisant à de "belles" propriétés, dont les quantifications d'algèbres enveloppantes d'algèbres de Lie semi-simples et d'algèbres de Hopf de coordonnées sont les exemples fondamentaux.

Des connections variées en mathématiques et en physique

La théorie des groupes quantiques a depuis lors été l'objet d'un grand intérêt et fascine par la diversité des connections, souvent surprenantes et non établies auparavant, qu'elle entretient avec des domaines des mathématiques tels que la théorie des noeuds et de 3-variétés, la théorie des représentations des algèbres de Lie en caractéristique positive ou encore la combinatoire avec la théorie des bases cristallines.

Cette première connection émerge en 1984 lorsque la théorie des noeuds aborde un virage déterminant : V. F. R. Jones [Jon85] découvre un invariant polynomial de noeuds grâce à des techniques totalement nouvelles faisant intervenir les groupes quantiques [1]. En effet, l'ingrédient principal de la construction de Jones est l'utilisation d'une solution R de l'*équation de Yang-Baxter quantique*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{YBE}$$

1. En fait, la construction originelle de V. F. R. Jones utilise les algèbres de Von Neumann. La reformulation de la théorie de Jones à l'aide des groupes quantiques est due à N. Reshetikhin et V. G. Turaev [RT90].

où $R \in \mathrm{End}(V \otimes V)$, V espace vectoriel de dimension finie, et $R_{12} = R \otimes \mathrm{Id} \in \mathrm{End}(V^{\otimes 3})$ (R_{23}, R_{13} étant définis de manière analogue). Une telle solution R est communément appelée R-matrice. En prenant une trace de Markov adaptée, on obtient un invariant de noeuds dépendant du paramètre de déformation q : c'est le polynôme de Jones. Les groupes quantiques sont dans cette théorie l'outil principal pour construire de façon systématique des solutions de l'équation (YBE). V. G. Drinfeld [Dri87] a ainsi obtenu des formules pour les R-matrices dans le cas sl_2. Le calcul des R-matrices dans le cas sl_{n+1} a été établi par M. Rosso [Ros89], et dans le cas général par A. N. Kirillov et N. Reshetikhin [KR90], S. Levendorskii et Y. Soibelman [LS90].

Pour les invariants sur les 3-variétés, nous pouvons citer les travaux de E. Witten [Wit89] provenant majoritairement de la physique, en terme de théorie des champs quantiques. À la lumière de ces résultats, N. Reshetikhin et V. G. Turaev [RT91] ont réalisé ces invariants à partir de représentations de groupes quantiques aux racines de l'unité. L'un des enjeux actuels est le développement d'invariants catégoriques pour les 3-variétés, dans l'esprit des travaux de M. Khovanov. Une étape difficile de ce programme est la catégorification de groupes quantiques aux racines de l'unité (à ce sujet, voir les travaux récents [Kho05, KQ12, EQ13]).

L'apport des groupes quantiques à l'étude des représentations des algèbres de Lie en caractéristique positive est tout aussi remarquable. Lorsque le paramètre de déformation q n'est pas une racine de l'unité, la théorie des représentations de l'algèbre de Lie simple complexe est essentiellement la même que pour le groupe quantique associé. Le cas où q est une racine de l'unité est très différent : par exemple la catégorie des représentations de dimension finie du groupe quantique n'est cette fois plus semi-simple. En fait, si q est une racine p-ème de l'unité, la théorie des représentations du groupe quantique est étroitement liée à celle de l'algèbre de Lie sous-jacente sur le corps fini $\mathbb{F}_p$ à p éléments. Les applications sont nombreuses notamment en théorie modulaire. Mentionnons également des connections avec la théorie des représentations des algèbres de Hecke aux racines de l'unité et son lien avec celle du groupe symétrique en caractéristique positive (voir [Jac12] à ce sujet).

Dans le contexte des modèles intégrables, le paramètre de quantification q représente la température et $q = 0$ correspond au zéro absolu. Il est alors raisonnable d'espérer que l'étude des groupes quantiques et de leurs représentations soit simplifiée dans cette situation. C'est en effet ce qu'a découvert M. Kashiwara en 1991 en introduisant la notion de bases cristallines [Kas91] : en $q = 0$, il existe une bonne base (dite base cristalline) pour ces représentations qui possède une structure combinatoire riche appelée graphe cristallin. Cet outil puissant permet de réduire de nombreuses propriétés des représentations du groupe quantique à de la combinatoire. À la même période, une approche plus géométrique due à G. Luzstig a également mené à la notion similaire de bases canoniques [Lus91].

De manière non-exhaustive, citons également des liens entre les groupes quantiques et les fonctions spéciales, la topologie en basse dimension, les algèbres d'opérateurs ou encore la géométrie non-commutative.

Les groupes quantiques ont une origine physique, dans le procédé de diffusion inverse quantique en théorie des modèles intégrables. Ils interviennent également dans des domaines variés tels que la physique des particules élémentaires ou encore la théorie des champs quantiques et des champs conformes. En effet on observe de façon expérimentale que les règles de fusion en théorie des champs conformes peuvent être reproduites en considérant des produits tensoriels de représentations de groupes quantiques aux racines de l'unité. Des résultats de T. Kohno [Koh89] et V. G. Drinfeld [Dri89], et ultérieurement de D. Kazhdan et G. Lusztig [KL93], viennent confirmer les liens profonds existant entre groupes quantiques et théorie des champs conformes.

Les algèbres affines quantiques

Définition de $\mathcal{U}_q(\hat{\mathfrak{g}})$

Soit $\mathfrak{g}$ une algèbre de Lie simple sur $\mathbb{C}$. Le groupe quantique associé $\mathcal{U}_q(\mathfrak{g})$ peut être considéré comme une généralisation de $\mathfrak{g}$ dans le sens où, en prenant la limite classique $q = 1$, on retrouve l'algèbre enveloppante universelle $\mathcal{U}(\mathfrak{g})$. Une autre algèbre a une importance toute particulière par ses propriétés et la variété de ses applications, il s'agit de l'algèbre de Lie affine $\hat{\mathfrak{g}}$ associée à $\mathfrak{g}$. C'est la généralisation naturelle en dimension infinie de l'algèbre de Lie $\mathfrak{g}$. L'un des faits remarquables des algèbres de Lie affines est qu'elles admettent deux réalisations : l'une donnée par les générateurs et relations usuels définissant une algèbre de Kac-Moody, l'autre comme extension centrale de l'algèbre de Lie des applications régulières de $\mathbb{C}^* \to \mathfrak{g}$.

Ces deux procédés (affinisation et quantification) peuvent alors être utilisés simultanément : à l'algèbre de Lie affine $\hat{\mathfrak{g}}$ est associée la déformation $\mathcal{U}_q(\hat{\mathfrak{g}})$ de son algèbre enveloppante universelle. C'est l'algèbre affine quantique, qu'on réalise ici à l'aide des générateurs de Serre - Chevalley. Comme dans le cas classique, les algèbres affines quantiques admettent une autre présentation en terme de générateurs de Drinfeld. En effet par un procédé d'affinisation due à V. G. Drinfeld, l'algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$ s'obtient à partir du groupe quantique $\mathcal{U}_q(\mathfrak{g})$ associé [Dri88, Bec94]. Ainsi les procédés d'affinisation et de quantification peuvent être effectués “dans n'importe quel ordre”, ce qui se traduit par la commutativité du diagramme suivant [EFK98]

$$\begin{tikzcd}
 & \mathcal{U}_q(\mathfrak{g}) \arrow[dr, "\text{Procédé d'affinisation de Drinfeld}"] & \\
\mathfrak{g} \arrow[ur, "\text{Quantification}"] \arrow[dr, "\text{Affinisation}"'] & & \mathcal{U}_q(\hat{\mathfrak{g}}) \\
 & \hat{\mathfrak{g}} \arrow[ur, "\text{Quantification}"'] &
\end{tikzcd}$$

Cette propriété, et beaucoup d'autres, font des algèbres affines quantiques un sujet de recherche extrêmement actif aux applications variées (voir par exemple [AK97, CP91, CP95, CP94, FR99, FM01, EM03, Nak01, Nak03]). En outre elles permettent, à partir d'une représentation irréductible de dimension finie V de $\mathcal{U}_q(\hat{\mathfrak{g}})$, la construction de solutions de l'équation de Yang-Baxter avec paramètres spectraux :

$$R_{12}(u)R_{13}(uv)R_{23}(v) = R_{23}(v)R_{13}(uv)R_{12}(u).$$

Ici l'inconnue $R(u)$ de cette équation est un endomorphisme de $V \otimes V$ qui dépend d'un paramètre $u \in \mathbb{C}^*$.

Représentations de dimension finie de $\mathcal{U}_q(\hat{\mathfrak{g}})$

Aussi l'étude des représentations de dimension finie des algèbres affines quantiques est-elle un enjeu considérable et est l'objet d'intenses recherches. Contrairement au cas semi-simple, les représentations de plus haut poids de $\mathcal{U}_q(\hat{\mathfrak{g}})$ sont de dimension infinie. En 1991,

V. Chari et A. Pressley [CP95] utilisent la réalisation de Drinfeld de $\mathcal{U}_q(\hat{\mathfrak{g}})$ pour obtenir une classification des représentations de dimension finie : ce sont des représentations dîtes de plus haut ℓ-poids [2], analogue pour la réalisation de Drinfeld des représentations de plus haut poids, paramétrées par des I_0-uplets de polynômes dont le coefficient constant est 1 (avec I_0 l'ensemble des sommets du diagramme de Dynkin associé à $\mathfrak{g}$).

Pour la réalisation de Drinfeld de l'algèbre affine quantique, le morphisme de caractère ne permet pas de comprendre correctement l'anneau de Grothendieck $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ des représentations de dimension finie. E. Frenkel et N. Reshetikhin [FR99] ont introduit en 1999 la notion de q–caractères : il s'agit d'un morphisme d'anneaux injectif

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \longrightarrow \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I_0, a \in \mathbb{C}^*}$$

décrivant l'action de la sous-algèbre de Cartan sur les représentations de $\mathcal{U}_q(\hat{\mathfrak{g}})$ de dimension finie (les monômes correspondant ici à des valeurs propres communes de cette sous-algèbre, appelés ℓ-poids). Il n'existe pas de formule générale pour le calcul du q–caractère d'une représentation. Cependant H. Nakajima [Nak01] a exprimé dans les cas ADE les q–caractères irréductibles en terme de cohomologie de certaines variétés de carquois.

Représentations extrémales de $\mathcal{U}_q(\hat{\mathfrak{g}})$

M. Kashiwara [Kas94, Kas02b] a défini une classe de représentations intégrables des algèbres affines quantiques [3] appelées représentations extrémales. Ce sont des représentations $V(\lambda)$, avec une base cristalline $\mathcal{B}(\lambda)$, paramétrées par λ dans P le réseau des poids intégraux de $\hat{\mathfrak{g}}$, et définies par des relations à partir d'un vecteur dit extrémal. Lorsque λ est dominant, on retrouve une représentation simple de plus haut poids. Mais en général $V(\lambda)$ n'est pas simple, n'est pas de plus haut poids ou de plus bas poids, ou même dans la catégorie $\mathcal{O}$. Ces représentations ont une importance particulière car, pour certaines valeurs de λ, elles ont des quotients de dimension finie. M. Kashiwara a ainsi démontré l'existence de bases cristallines pour les représentations fondamentales de dimension finie de $\mathcal{U}_q(\hat{\mathfrak{g}})$ (pour des paramètres spectraux particuliers).

Connections avec d'autres domaines des mathématiques

Les algèbres affines quantiques apparaissent dans bien d'autres domaines des mathématiques. En type A, elles sont en dualité de Froebenius-Schur avec les algèbres de Hecke affines [CP94]. D. Hernandez et B. Leclerc [HL10] ont également démontré qu'il existe une structure d'algèbre amassée sur des sous-anneaux de l'anneau de Grothendieck $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ des représentations de dimension finie. Cette catégorification monoïdale des algèbres amassées a permis de résoudre des conjectures de positivité.

2. La lettre ℓ est communément utilisée en théorie des représentations de l'affinisation quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$ d'une algèbre de Kac-Moody quantique $\mathcal{U}_q(\mathfrak{g})$. Elle provient du mot lacet (loop en anglais) et rappelle qu'il est ici question de l'analogue quantique de l'algèbre de Lie des lacets de $\mathbb{C}^* \to \mathfrak{g}$ associée à $\mathfrak{g}$. Notons que le cas des algèbres affines quantiques $\mathcal{U}_q(\hat{\mathfrak{g}})$ est singulier : ces algèbres ayant deux réalisations en terme de générateurs de Serre - Chevalley et en terme de générateurs de Drinfeld, il y a pour $\mathcal{U}_q(\hat{\mathfrak{g}})$ les notions de représentations de plus haut poids et de plus haut ℓ-poids. En particulier, la présence de la lettre ℓ est essentielle dans ce cas.

3. On réalise ici $\mathcal{U}_q(\hat{\mathfrak{g}})$ à l'aide des générateurs de Serre - Chevalley.

Les algèbres toroïdales quantiques $\mathcal{U}_q(\mathfrak{g}^{tor})$

Définition de $\mathcal{U}_q(\mathfrak{g}^{tor})$

Maintenant, les algèbres affines quantiques étant en particulier des algèbres de Kac-Moody quantiques, il est possible de les affiniser de nouveau par le procédé de V. G. Drinfeld [Dri88] (voir le diagramme ci-dessous [EFK98]). L'algèbre ainsi obtenue est appelée l'algèbre toroïdale quantique (ou algèbre doublement affine) et notée $\mathcal{U}_q(\mathfrak{g}^{tor})$.

$$\begin{tikzcd}
 & \mathcal{U}_q(\mathfrak{g}) \arrow[dr, "\text{Affinisation}"] & & \\
\mathfrak{g} \arrow[ur, "\text{Quantification}"] \arrow[dr, "\text{Affinisation}"'] & & \mathcal{U}_q(\hat{\mathfrak{g}}) \arrow[dr, "\text{Affinisation}"] & \\
 & \hat{\mathfrak{g}} \arrow[ur, "\text{Quantification}"'] & & \mathcal{U}_q(\mathfrak{g}^{tor})
\end{tikzcd}$$

Ce procédé ne peut alors être répété, l'algèbre toroïdale quantique n'étant pas pour sa part une algèbre de Kac-Mooky quantique. Ainsi $\mathcal{U}_q(\mathfrak{g}^{tor})$ apparait dans cette théorie comme l' “objet terminal” et la généralisation “ultime” de l'algèbre de Lie $\mathfrak{g}$ et du groupe quantique associé $\mathcal{U}_q(\mathfrak{g})$. De plus, les deux réalisations de l'algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$ se retrouvent comme sous-algèbres de l'algèbre toroïdale : comme sous-algèbre affine quantique horizontale $\mathcal{U}_q^h(\mathfrak{g}^{tor})$ et comme sous-algèbre affine quantique verticale $\mathcal{U}_q^{v,0}(\mathfrak{g}^{tor})$.

Les algèbres toroïdales quantiques ont été introduites pour la première fois en type A par V. Ginzburg, M. Kapranov et É. Vasserot en 1995 [GKV95], et de façon générale par N. Jing [Jin98] et H. Nakajima [Nak01]. Dans le cas des diagrammes de Dynkin simplement lacés, une construction géométrique des algèbres toroïdales quantiques qui utilise les variétés de carquois a été développée par H. Nakajima [Nak02] (voir aussi [VV]). Comme pour les algèbres affines quantiques en type A, elles sont en dualité avec les algèbres de Cherednik elliptiques [VV96] (algèbres de Hecke doublement affines elliptiques). L'étude des représentations de ces algèbres n'en est qu'à ses débuts. Réputée difficile (voir l'introduction de [FJMM13]), elle est encore très peu comprise. Mais on peut en espérer des applications prometteuses et variées, comme c'est le cas pour les algèbres affines quantiques et les algèbres de Cherednik [Che95].

Représentations de $\mathcal{U}_q(\mathfrak{g}^{tor})$

Les représentations irréductibles intégrables des algèbres toroïdales quantiques dans la catégorie $\mathcal{O}$ ont été classifiées en type A par K. Miki en 2000 [Mik00]. Ces résultats ont successivement été étendus par H. Nakajima [Nak01] puis D. Hernandez [Her05]. Ces représentations sont, comme pour l'algèbre affine quantique, des représentations de plus haut ℓ-poids paramétrées par des I-uplets de polynômes dont le coefficient constant est 1 (où I est l'ensemble des sommets du Diagramme de Dynkin de $\hat{\mathfrak{g}}$). Dans le cas de $\mathcal{U}_q(\mathfrak{g}^{tor})$, elles sont de dimension infinie. Il existe pour ces algèbres aussi une théorie des q–caractères développée par D. Hernandez [Her05], analogue au cas des algèbres affines quantiques. L'une des premières réussites de cette théorie est la construction d'une action de l'algèbre toroïdale quantique de type A sur l'espace de q–Fock [VV98, STU98].

Plus récemment, B. Feigin, M. Jimbo, T. Miwa et E. Mukhin [FJMM13] ont construit des représentations de la d-déformation de l'algèbre toroïdale quantique de type A ayant pour bases naturelles des partitions planes (avec certaines conditions supplémentaires). Par restriction, ils obtiennent une description combinatoire des modules de type Weyl sur l'algèbre affine quantique utilisés en théorie des représentations des W-algèbres. Des applications sont également espérées pour les constructions de type Wakimoto des modules de Feigin-Fuchs sur l'algèbre affine quantique.

Représentations ℓ-extrémales de $\mathcal{U}_q(\mathfrak{g}^{tor})$

L'une des questions essentielles de la théorie des représentations des algèbres toroïdales quantiques est la construction de représentations de dimension finie [4]. En effet pour le moment, très peu d'exemples de telles représentations existent [5]. Dans l'esprit des travaux de M. Kashiwara sur les représentations extrémales, D. Hernandez [Her09] a proposé la définition de représentations ℓ-extrémales pour les algèbres toroïdales quantiques.

Définition. *[Her09] Une représentation ℓ-extrémale de l'algèbre toroïdale quantique $\mathcal{U}_q(\mathfrak{g}^{tor})$ de ℓ-poids m est une représentation intégrable V telle qu'il existe un vecteur v de ℓ-poids m satisfaisant à*

(i) $\mathcal{U}_q(\mathfrak{g}^{tor}) \cdot v = V$,

(ii) v est extrémal pour la sous-algèbre affine quantique horizontale $\mathcal{U}_q^h(\mathfrak{g}^{tor})$ au sens de M. Kashiwara [Kas94],

(iii) les sous-algèbres verticales $\mathcal{U}_q^{v,i}(\mathfrak{g}^{tor})$ agissent de façon localement finie sur V.

Comme dans la théorie de M. Kashiwara, l'une des motivations principales de cette définition est la construction de représentations de dimension finie, cette fois-ci aux racines de l'unité. Jusqu'alors, très peu de telles représentations étaient connues. En particulier, il n'existait pas de procédé général de construction de représentations ℓ-extrémales ou encore de représentations de dimension finie de l'algèbre toroïdale aux racines de l'unité.

4. Les représentations de dimension finie obtenues dans cette thèse le sont par spécialisation du paramètre quantique à une racine de l'unité. Aucun exemple de représentation de dimension finie n'est connu dans le cas où q est générique. En particulier, le procédé proposé ici nécessite cette spécialisation aux racines de l'unité.

5. Le premier exemple d'une telle représentation est donné dans [Her09]. Exception faite des représentations construites dans cette thèse, il n'y en a pas d'autre à notre connaissance.

Résultats obtenus dans cette thèse

Cette thèse s'inscrit dans cette direction. Nous présentons trois procédés de construction de représentations ℓ-extrémales pour l'algèbre toroïdale quantique $\mathcal{U}_q(sl_{n+1}^{tor})$ $(n \geq 2)$ de type A :

1. via les cristaux monomiaux de M. Kashiwara et de H. Nakajima (Chapitre 2 - article [Man12c] accepté pour publication dans *Pacific Journal of Mathematics*),

2. par produits de fusion via le coproduit de Drinfeld (Chapitre 3 - article [Man13a] prépublié arXiv:1305.3481),

3. via l'algèbre affinisée quantique $\mathcal{U}_q(\hat{sl}_\infty)$ (Chapitre 4 - article [Man13b] en préparation).

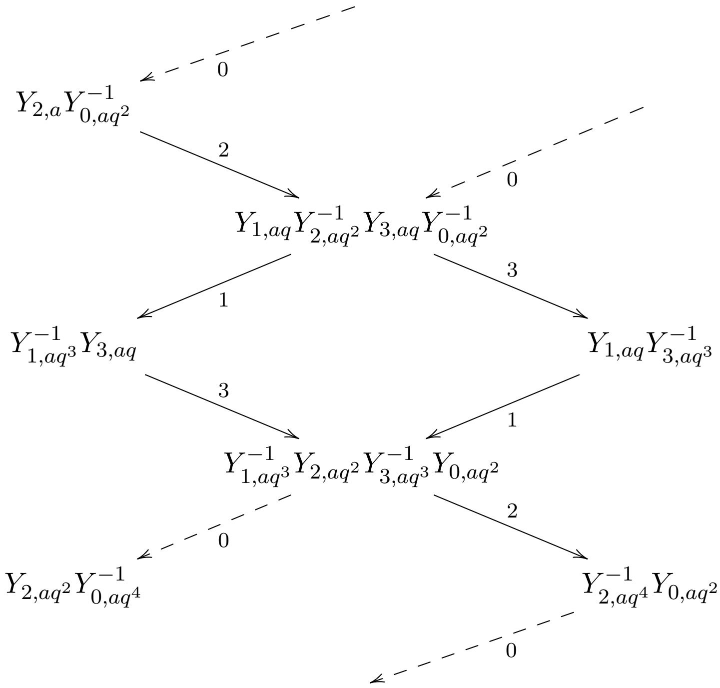
Nous construisons ainsi de nouvelles familles de représentations ℓ-extrémales. En spécialisant le paramètre quantique, nous obtenons de nouvelles représentations de dimension finie de l'algèbre toroïdale quantique aux racines de l'unité.

Durant la préparation de cette thèse, il est apparu que certaines des représentations que nous avons construites (les *représentations vectorielles*) sont également utilisées par B. Feigin, M. Jimbo, T. Miwa et E. Mukhin [FJMM13] dans le cas de la d-déformation $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ de l'algèbre toroïdale quantique de type A [6]. Notons cependant que la construction des représentations vectorielles y est très différente : elles sont obtenues dans [FJMM13] comme déformation quantique d'un module sur une algèbre de Lie d'opérateurs différentiels. Comme nous l'avons déjà évoqué, leurs motivations sont également distinctes des nôtres, avec des perspectives en théorie des modules de type Weyl et de Feigin-Fuchs (voir [FJMM13]).

Construction via les cristaux monomiaux

Le procédé de construction des représentations ℓ-extrémales présenté dans le chapitre 2 de cette thèse repose sur le cristal monomial $\mathcal{M}_I$ définit par M. Kashiwara et H. Nakajima [Kas03, Nak03]. Les sommets de ce cristal sont des monômes dans $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*}$, et les opérateurs de Kashiwara sont définis par analogie avec la combinatoire des q-caractères d'une algèbre affinisée quantique. Il permet de réaliser les cristaux de type fini pour le groupe quantique $\mathcal{U}_q(sl_{n+1})$. En 2006, D. Hernandez et H. Nakajima [HN06] obtiennent une réalisation monomiale des bases cristallines $\mathcal{B}(\varpi_\ell)$ $(\ell \in I_0)$ des représentations extrémales fondamentales $V(\varpi_\ell)$ $(\varpi_\ell = \Lambda_\ell - \Lambda_0$, Λ_i i-ème poids fondamental) de l'algèbre affine quantique $\mathcal{U}_q(\hat{sl}_{n+1})$ lorsque $n = 2r + 1$ est supposé impair : le cristal $\mathcal{B}(\varpi_\ell)$ est isomorphe au sous-cristal $\mathcal{M}_{\ell,a}$ de $\mathcal{M}_I$ engendré par le monôme $Y_{\ell,a} Y_{0,aq^{d_\ell}}^{-1}$ (avec $a \in \mathbb{C}^*$ et d_ℓ la distance des sommets 0 et ℓ dans le diagramme de Dynkin). Par exemple, le cristal $\mathcal{M}_{2,a}$ pour l'algèbre affine quantique $\mathcal{U}_q(\hat{sl}_4)$ est

6. Nous voudrions remercier H. Nakajima pour nous avoir fait remarquer ce point.



Comme nous l'avons déjà évoqué, ces monômes se retrouvent dans la théorie des q-caractères de l'algèbre toroïdale quantique $\mathcal{U}_q(sl^{tor}_{n+1})$. D'où une question naturelle : existe-t-il une représentation $V_{\ell,a}$ de $\mathcal{U}_q(sl^{tor}_{n+1})$ dont le q-caractère serait la somme des monômes apparaissant dans le cristal $\mathcal{M}_{\ell,a}$? Si c'est le cas cette représentation serait un bon candidat pour être une représentation ℓ-extrémale. Seulement, une condition nécessaire est que cet ensemble de monômes satisfasse, en plus de sa structure de cristal, les propriétés combinatoires des q-caractères (voir [FM01]). Ceci nous amène à introduire dans cette thèse la notion d'ensemble de monômes clos : c'est un ensemble qui respecte les propriétés combinatoires du cristal et des q-caractères. Dès lors pour que le cristal $\mathcal{M}_{\ell,a}$ soit clos, des conditions restrictives sur le paramètre $\ell \in I_0$ doivent être prises. En effet, on a

Proposition. *(Theorem 2.2.22) Supposons $n = 2r+1$ impair ($r \geq 1$). Le cristal monomial $\mathcal{M}_{\ell,a}$ est clos si et seulement si*

$$\ell = 1, r+1 \ \ ou \ n. \tag{C1}$$

Ce résultat s'obtient par étude des symétries des cristaux $\mathcal{M}_{\ell,a}$. En effet, le diagramme de Dynkin de type $A^{(1)}_n$ est cyclique. L'automorphisme de rotation se retrouve au niveau des cristaux par l'existence d'opérateurs de promotion. Ces opérateurs ont été introduit par M. Shimozono en 2002 dans l'étude des cristaux de type fini sur $\mathcal{U}_q(sl_{n+1})$ [Shi02]. On introduit ici la notion d'opérateurs de promotion pour les cristaux extrémaux fondamentaux sur $\mathcal{U}_q(\hat{sl}_{n+1})$ qui traduit les symétries dans $\mathcal{M}_{\ell,a}$ (voir [Man12a] pour le cas du $\mathcal{U}_q(\hat{sl}_4)$-cristal $\mathcal{M}_{2,a}$). Leur rôle est déterminant dans la démonstration de cette proposition.

Il est donc nécessaire de supposer que ℓ satisfasse (C1) pour espérer construire une représentation $V_{\ell,a}$ de $\mathcal{U}_q(sl^{tor}_{n+1})$ dont le q-caractère est la somme des monômes apparaissant dans le cristal $\mathcal{M}_{\ell,a}$. En fait cette condition est suffisante.

Proposition. *(Theorem 2.3.1) Supposons que $\ell \in I_0$ satisfasse $(C1)$. Il existe une représentation $V_{\ell,a}$ de $\mathcal{U}_q(sl_{n+1}^{tor})$ telle que*

$$\chi_q(V_{\ell,a}) = \sum_{m \in \mathcal{M}_{\ell,a}} m.$$

Pour construire les représentations $V_{\ell,a}$, on “colle” ensemble des représentations de dimension finie des sous-algèbres affines quantiques verticales (voir [Man12a]). Notons qu'ici aussi les opérateurs de promotion jouent un rôle primordial dans notre construction, puisqu'ils permettent de s'assurer de la compatibilité entre les actions des différentes algèbres verticales. Cette construction des représentations $V_{\ell,a}$ s'étend au cas $n \geq 2$ pair et $\ell = 1, n$.

Rappelons que le but était de construire des représentations ℓ-extrémales. On a le

Théorème. *(Theorem 2.3.7) Supposons que $\ell \in I_0$ satisfasse $(C1)$. La représentation $V_{\ell,a}$ de $\mathcal{U}_q(sl_{n+1}^{tor})$ est une représentation ℓ-extrémale associée au ℓ-poids $Y_{\ell,a}Y^{-1}_{0,aq^{d_\ell}}$. On l'appelle la représentation ℓ-extrémale fondamentale.*

Lorsque le cristal $\mathcal{M}_{\ell,a}$ n'est pas clos, il n'est pas possible de lui associer une représentation de l'algèbre toroïdale quantique de façon directe. On peut néanmoins procéder comme ceci : considérer à la place de $\mathcal{M}_{\ell,a}$ un cristal $\overline{\mathcal{M}}_{\ell,a}$ le contenant et qui soit clos. La question de l'existence d'une représentation, cette fois-ci associée à $\overline{\mathcal{M}}_{\ell,a}$, se pose de nouveau. Cependant la structure de $\overline{\mathcal{M}}_{\ell,a}$ est plus compliquée que celle de $\mathcal{M}_{\ell,a}$, et il est difficile (pour nous en tout cas) de construire systématiquement l'éventuelle représentation de $\mathcal{U}_q(sl_{n+1}^{tor})$ associée à $\overline{\mathcal{M}}_{\ell,a}$. On se propose donc de vérifier sur un exemple qu'une telle représentation existe bien.

En plus d'opérateurs de promotion, il existe un automorphisme de décalage du paramètre spectral

$$\tau_{p_\ell} : Y^{\pm 1}_{i,aq^s} \longrightarrow Y^{\pm 1}_{i,aq^{s+p_\ell}}$$

sur les cristaux $\mathcal{M}_{\ell,a}$. Par exemple pour le $\mathcal{U}_q(\hat{sl}_4)$-cristal $\mathcal{M}_{2,a}$, τ_2 induit un automorphisme de $\mathcal{M}_{2,a}$. Ces automorphismes de décalage sont reliés à des représentations de dimension finie de l'algèbre toroïdale quantique aux racines de l'unité. Plus précisément en spécialisant les relations définissant les représentations ℓ-extrémales fondamentales aux racines de l'unité, on obtient le

Théorème. *(Theorem 2.3.18) Supposons que ℓ satisfasse $(C1)$. Soit $L \geq 1$ et ϵ une racine primitive $(p_\ell L)$ de l'unité. Il existe une représentation irréductible $[V_{\ell,a}]_\epsilon$ de l'algèbre toroïdale $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$ de dimension finie égale à $L \times \begin{pmatrix} n+1 \\ \ell \end{pmatrix}$.*

Construction par produits de fusion

Dans le chapitre 3 de cette thèse, nous présentons une construction des représentations ℓ-extrémales par produits de fusion [7] pour l'algèbre toroïdale quantique $\mathcal{U}_q(sl_{n+1}^{tor})$. On s'inspire ici des travaux de M. Kashiwara sur les représentations extrémales des algèbres affines quantiques : elles sont étroitement liées au produit tensoriel d'un module de plus

7. Le terme de produit de fusion est utilisé dans [Her07] dans le cadre de produits tensoriels de représentations d'algèbres affinisées quantiques appartenant à la catégorie $\mathcal{O}_{\mathrm{int}}$. Nos travaux généralisant ces constructions, nous reprenons cette terminologie ici.

haut poids par un module de plus bas poids. Dans cet esprit, on étudie ici le produit tensoriel ($\ell \in I_0$ et $a \in \mathbb{C}^*$)

$$V(Y_{\ell,a}) \otimes V(Y^{-1}_{0,aq^{d_\ell}})$$

d'un module fondamental simple de plus haut ℓ-poids $Y_{\ell,a}$ par un module fondamental simple de plus bas ℓ-poids $Y^{-1}_{0,aq^{d_\ell}}$. Il n'y a pas de structure d'algèbre de Hopf connue sur l'algèbre toroïdale quantique [8]. Cependant il existe un coproduit Δ_D (le coproduit de Drinfeld) à valeur dans une complétion du produit tensoriel. Ce coproduit, sous réserve de résultats de convergence, peut être utilisé pour définir une action de l'algèbre toroïdale quantique. Citons notamment à ce sujet les travaux récents de B. Feigin, M. Jimbo, T. Miwa et E. Mukhin [FFJ$^+$11a, FFJ$^+$11b, FJMM12, FJMM13] ou de D. Hernandez [Her07].

Nous introduisons ici un procédé pour définir, à l'aide du coproduit de Drinfeld, une action de l'algèbre toroïdale quantique sur $V(Y_{\ell,a}) \otimes V(Y^{-1}_{0,aq^{d_\ell}})$ (en fait, ce procédé permet de définir une action de l'algèbre toroïdale quantique sur une classe plus importante de produits tensoriels) :

Théorème. *(Theorem 3.2.1) Soit $\ell \in I_0$ et $a \in \mathbb{C}^*$. Le coproduit Δ_D est bien défini sur*

$$V(Y_{\ell,a}) \otimes V(Y^{-1}_{0,aq^{d_\ell}})$$

et munit ce produit tensoriel d'une structure de module sur l'algèbre toroïdale quantique $\mathcal{U}_q(sl^{tor}_{n+1})$.

Considérons $T_{\ell,a}$ le sous-module de $V(Y_{\ell,a}) \otimes V(Y^{-1}_{0,aq^{d_\ell}})$ engendré par le produit tensoriel d'un vecteur de plus haut ℓ-poids par un vecteur de plus bas ℓ-poids. Il constitue notre candidat pour être une représentation ℓ-extrémale de ℓ-poids $Y_{\ell,a}Y^{-1}_{0,aq^{d_\ell}}$. En fait, la cyclicité du diagramme de Dynkin nous impose les conditions suivantes : pour que $T_{\ell,a}$ soit ℓ-extrémale, il faut que n et ℓ satisfassent à la condition

$$(\ell = 1, n) \text{ ou } (n = 2r + 1 \text{ et } \ell = r + 1). \tag{C2}$$

On conjecture que cette condition est également suffisante.

Conjecture. *(Conjecture 3.2.13) Supposons que n et ℓ satisfassent $(C2)$. Alors $T_{\ell,a}$ est une représentation ℓ-extrémale de ℓ-poids $Y_{\ell,a}Y^{-1}_{0,aq^{d_\ell}}$, isomorphe à la représentation ℓ-extrémale fondamentale $V_{\ell,a}$.*

On démontre cette conjecture pour $\ell = 1, n$. Lorsque $\ell = 1$, la représentation fondamentale ℓ-extrémale $V_{1,a}$ est également appelée *représentation vectorielle* de $\mathcal{U}_q(sl^{tor}_{n+1})$.

On étudie le produit tensoriel de représentations ℓ-extrémales fondamentales lorsque les paramètres spectraux sont choisis génériquement. On obtient de cette manière de nouvelles représentations ℓ-extrémales. En effet on a le

Théorème. *(Theorem 3.3.2, Theorem 3.4.3) Supposons que n et ℓ satisfassent $(C2)$. Soient $k \in \mathbb{N}^*$ et $a_1, a_2, \cdots, a_k \in \mathbb{C}^*$ tels que*

$$\frac{a_i}{a_j} \notin q^{\pm 2 + p_\ell \mathbb{Z}} \text{ pour tout } i < j.$$

Alors le coproduit Δ_D munit le produit tensoriel

$$V_{\ell,a_1} \otimes V_{\ell,a_2} \otimes \cdots \otimes V_{\ell,a_k}$$

8. Voir cependant les résultats annoncés dans *Yangians for Kac-Moody algebras* par N. Guay et H. Nakajima.

d'une structure de représentation irréductible sur $\mathcal{U}_q(sl_{n+1}^{tor})$ qui est ℓ-extrémale de ℓ-poids

$$Y_{\ell,a_1}\cdots Y_{\ell,a_k}Y_{0,a_1q^{d_\ell}}^{-1}\cdots Y_{0,a_1q^{d_\ell}}^{-1}.$$

On considère en particulier le cas de produits tensoriels de représentations vectorielles lorsque l'ensemble des paramètres spectraux forme un q-segment. On obtient de cette manière une autre famille de représentations ℓ-extrémales.

Proposition. *(Proposition 3.3.7) Soit $k \in \mathbb{N}^*$ et $a \in \mathbb{C}^*$. Il existe un sous-espace vectoriel $V_{1,\boldsymbol{a}}$ (où $\boldsymbol{a} = \{a, aq^{-2}, \cdots, aq^{-2(k-1)}\}$) de*

$$V_{1,a}\otimes V_{1,aq^{-2}}\otimes\cdots\otimes V_{1,aq^{-2(k-1)}}$$

qui peut être muni d'une structure de représentation sur l'algèbre toroïdale quantique $\mathcal{U}_q(sl_{n+1}^{tor})$. De plus, $V_{1,\boldsymbol{a}}$ est irréductible et ℓ-extrémale de ℓ-poids

$$Y_{1,a}Y_{1,aq^{-2}}\cdots Y_{1,aq^{-2(k-1)}}Y_{0,aq}^{-1}Y_{0,aq^{-1}}^{-1}\cdots Y_{0,aq^{-2(k-1)+1}}^{-1}.$$

On retrouve également toutes les représentations ℓ-extrémales fondamentales obtenues dans la première partie à partir de ce produit tensoriel. En effet, on a le

Théorème. *(Theorem 3.3.13) Supposons $n = 2r+1$ impair ($r \geq 1$). Le coproduit Δ_D munit le produit tensoriel*

$$V_{1,a}\otimes V_{1,aq^{-2}}\otimes\cdots\otimes V_{1,aq^{-2r}}$$

d'une structure de représentation sur $\mathcal{U}_q(sl_{n+1}^{tor})$. Elle est non irréductible, et admet un sous-module isomorphe à la représentation ℓ-extrémale fondamentale $V_{r+1,aq^{-r-2}}$.

Par spécialisation du paramètre quantique, on obtient de nouvelles représentations de dimension finie de l'algèbre toroïdale aux racines de l'unité.

Théorème. *(Theorem 3.3.15, Theorem 3.4.4) Supposons que n et ℓ satisfassent ($C2$). Soient $L \geq 1$ et ϵ une racine primitive ($p_\ell L$) de l'unité. Soient également $k \in \mathbb{N}^*$ et $a_1, a_2, \cdots, a_k \in \mathbb{C}^*$ tels que*

$$\frac{a_i}{a_j} \notin \epsilon^{\pm 2+p_\ell\mathbb{Z}} \text{ pour tout } i<j.$$

Alors le coproduit Δ_D munit le produit tensoriel

$$[V_{\ell,a_1}]_\epsilon\otimes[V_{\ell,a_2}]_\epsilon\otimes\cdots\otimes[V_{\ell,a_k}]_\epsilon$$

d'une structure de représentation irréductible sur l'algèbre toroïdale $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$ de dimension finie $\left[\begin{pmatrix} n+1 \\ \ell \end{pmatrix}\times L\right]^k$.

On obtient un résultat analogue pour le produit tensoriel de représentations vectorielles lorsque l'ensemble des paramètres spectraux forme un q-segment.

Construction via l'algèbre affinisée quantique $\mathcal{U}_q(\hat{sl}_\infty)$

Dans le chapitre 4 de cette thèse, nous développons la théorie des représentations ℓ-extrémales pour l'algèbre affinisée $\mathcal{U}_q(\hat{sl}_\infty)$. Nos motivations pour l'étude de ces représentations sont doubles.

- La théorie des représentations de l'algèbre affinisée $\mathcal{U}_q(\hat{sl}_\infty)$ et de l'algèbre toroïdale quantique $\mathcal{U}_q(sl_{n+1}^{tor})$ $(n \geq 2)$ ont un lien combinatoire. En effet, le diagramme de Dynkin associé à l'algèbre $\mathcal{U}_q(sl_{n+1}^{tor})$ se déduit du diagramme de Dynkin de type A_∞ par passage au quotient par l'automorphisme de décalage de $n+1$ sommets. Cet automorphisme induit en particulier un morphisme d'anneau

$$\phi_n : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in \mathbb{Z}, a \in \mathbb{C}^*} \longrightarrow \mathbb{Z}[Y_{\bar{i},a}^{\pm 1}]_{i \in \mathbb{Z}/(n+1)\mathbb{Z}, a \in \mathbb{C}^*}$$

  donné par $\phi_n(Y_{i,a}^{\pm 1}) = Y_{\bar{i},a}^{\pm 1}$. Dans un article de 2011, D. Hernandez [Her11] conjecture que l'image par ϕ_n du q-caractère d'une représentation irréductible dans la catégorie $\mathcal{O}_{\mathrm{int}}$ est le q-caractère d'une "vraie" représentation de l'algèbre toroïdale quantique $\mathcal{U}_q(sl_{n+1}^{tor})$. De plus, il démontre que cette propriété est satisfaite par l'ensemble des modules de Kirillov-Reshetikhin de $\mathcal{U}_q(\hat{sl}_\infty)$. Dès lors, notre motivation est la suivante : construire des représentations ℓ-extrémales pour l'algèbre affinisée $\mathcal{U}_q(\hat{sl}_\infty)$ et déterminer l'image de leur q-caractère par ϕ_n. Nous verrons que c'est effectivement le q-caractère d'une représentation de $\mathcal{U}_q(sl_{n+1}^{tor})$, et qu'on retrouve de cette manière les représentations ℓ-extrémales fondamentales de l'algèbre toroïdale quantique. En particulier, nous démontrons la conjecture énoncée par D. Hernandez dans [Her11] pour les représentations ℓ-extrémales fondamentales de $\mathcal{U}_q(\hat{sl}_\infty)$.
- L'algèbre $\mathcal{U}_q(\hat{sl}_\infty)$ est définie comme la limite inductive des algèbres affines quantiques $\mathcal{U}_q(\hat{sl}_{n+1})$ $(n \geq 2)$. Pour cette raison, la théorie des représentations de $\mathcal{U}_q(\hat{sl}_\infty)$ est mieux comprise que pour une algèbre affinisée générale (en particulier mieux que pour les algèbres toroïdales quantiques) : les modules de Kirillov-Reshetikhin sont minces (i.e. l'action de la sous-algèbre de Cartan est diagonale et les sous-espaces propres associés sont tous de dimension un) et possèdent une base indexée par des tableaux de Young sur laquelle l'action de $\mathcal{U}_q(\hat{sl}_\infty)$ est explicitement connue. La situation est bien différente dans le cas des algèbres toroïdales quantiques : les modules fondamentaux ne sont pas minces en général et l'action de la sous-algèbre de Cartan pas diagonale. C'est le sens de l'étude faite dans [Her11] : utiliser la théorie des représentations de l'algèbre $\mathcal{U}_q(\hat{sl}_\infty)$ (plus simple) et le lien combinatoire précisé plus haut pour obtenir des formules de q-caractère pour des représentations de $\mathcal{U}_q(sl_{n+1}^{tor})$. C'est aussi la direction qu'on propose de prendre ici, étudier les représentations ℓ-extrémales de l'algèbre $\mathcal{U}_q(\hat{sl}_\infty)$ (qui sera plus aisée que dans le cas toroïdal) en espérant par le lien combinatoire construire des représentations ℓ-extrémales cette fois sur l'algèbre toroïdale quantique.

On construit des représentations ℓ-extrémales sur l'algèbre affinisée $\mathcal{U}_q(\hat{sl}_\infty)$ par produit de fusion, en reprenant les méthodes mises au point dans le chapitre 3 pour l'algèbre toroïdale. On considère le produit tensoriel ($\ell \geq 1$ et $a \in \mathbb{C}^*$)

$$V^\infty(Y_{\ell,a}) \otimes V^\infty(Y_{0,aq^{d_\ell}}^{-1})$$

d'un module fondamental simple de plus haut ℓ-poids $Y_{\ell,a}$ par un module fondamental simple de plus bas ℓ-poids $Y_{0,aq^{d_\ell}}^{-1}$. On définit sur ce produit tensoriel une action de $\mathcal{U}_q(\hat{sl}_\infty)$ grâce au coproduit de Drinfeld.

Proposition. *(Proposition 4.3.2) Le coproduit Δ_D est bien défini sur*

$$V^\infty(Y_{\ell,a}) \otimes V^\infty(Y^{-1}_{0,aq^{d_\ell}})$$

et munit ce produit tensoriel d'une structure de module sur l'algèbre affinisée $\mathcal{U}_q(\hat{sl}_\infty)$.

De même que dans le cas toroïdal, on considère $V^\infty_{\ell,a}$ le sous-module de $V^\infty(Y_{\ell,a}) \otimes V^\infty(Y^{-1}_{0,aq^{d_\ell}})$ engendré par le produit tensoriel d'un vecteur de plus haut ℓ-poids par un vecteur de plus bas ℓ-poids. On détermine une base de $V^\infty_{\ell,a}$ indexée par les tableaux de Young de forme (ℓ) sur laquelle l'action de $\mathcal{U}_q(\hat{sl}_\infty)$ est explicitement connue. En particulier on obtient le

Théorème. *(Theorem 4.3.14) Soit $\ell \geq 1$ et $a \in \mathbb{C}^*$. La représentation $V^\infty_{\ell,a}$ de $\mathcal{U}_q(\hat{sl}_\infty)$ est irréductible, mince et ℓ-extrémale de ℓ-poids $Y_{\ell,a}Y^{-1}_{0,aq^{d_\ell}}$. On l'appelle la représentation ℓ-extrémale fondamentale.*

Notons que les représentations ℓ-extrémales fondamentales peuvent également être obtenues grâce aux cristaux monomiaux, dans l'esprit du chapitre 2. La représentation $V^\infty_{1,a}$ est ici aussi appelée la représentation vectorielle de $\mathcal{U}_q(\hat{sl}_\infty)$. On obtient des résultats analogues au cas toroïdal par produit de fusion de représentations ℓ-extrémales fondamentales. En particulier, on retrouve toutes les représentations ℓ-extrémales fondamentales à partir d'un produit de fusion de représentations vectorielles.

La motivation principale de l'étude des représentations ℓ-extrémales de l'algèbre affinisée $\mathcal{U}_q(\hat{sl}_\infty)$ est son application au cas toroïdal. On a le

Théorème. *(Theorem 4.5.3) Supposons que $n \geq 2$ est pris tel que*

$$(n \text{ est pair et } \ell \leq n+1) \text{ ou } \left(n \text{ est impair et } \ell \leq \frac{n+1}{2}\right).$$

Alors $\phi_n\left(\chi_q(V^\infty_{\ell,a})\right)$ est le q-caractère d'une représentation de l'algèbre toroïdale quantique $\mathcal{U}_q(sl^{tor}_{n+1})$. On la note $[V^\infty_{\ell,a}]_n$.

En particulier, ce résultat confirme la conjecture énoncée [Her11] pour les représentations ℓ-extrémales fondamentales de $\mathcal{U}_q(\hat{sl}_\infty)$.

Rappelons que le but est la construction de représentations ℓ-extrémales pour l'algèbre toroïdale quantique $\mathcal{U}_q(sl^{tor}_{n+1})$. On obtient le résultat suivant dans ce sens.

Théorème. *(Theorem 4.5.4)*
- *(i) Supposons que $n \geq 2$. Alors $[V^\infty_{1,a}]_n$ est une représentation ℓ-extrémale de l'algèbre toroïdale quantique $\mathcal{U}_q(sl^{tor}_{n+1})$ isomorphe à la représentation vectorielle $V_{1,a}$.*
- *(ii) Supposons que $n = 2r+1$ est impair $(r \geq 1)$. Alors $[V^\infty_{r+1,a}]_n$ a un quotient irréductible isomorphe à la représentation ℓ-extrémale fondamentale $V_{r+1,a}$ de $\mathcal{U}_q(sl^{tor}_{n+1})$.*

Perspectives de nos travaux

Dans le chapitre 5 de cette thèse, on applique les procédés développés dans les chapitres 2 et 3 pour l'algèbre toroïdale quantique $\mathcal{U}_q(\mathfrak{g}^{tor})$ de type D_4 : on construit une représentation ℓ-extrémale analogue aux représentations vectorielles introduites pour $\mathcal{U}_q(sl^{tor}_{n+1})$ et $\mathcal{U}_q(\hat{sl}_\infty)$. On propose ici deux constructions de cette représentation, par produit de fusion ou en utilisant les réalisations monomiales. C'est le premier exemple d'une représentation

ℓ-extrémale en type différent du type A. De plus par spécialisation du paramètre quantique, on obtient des représentations irréductibles de dimension finie de l'algèbre toroïdale quantique de type D_4 aux racines de l'unité.

Dès lors, on peut raisonnablement espérer construire des représentations ℓ-extrémales pour les algèbres affinisées générales (en particulier pour toutes les algèbres toroïdales quantiques) par les procédés présentés dans les chapitres 2 et 3. Cela déboucherait sur la construction systématique de représentations de dimension finie aux racines de l'unité. En type A, on obtient par dualité des représentations de dimension finie pour l'algèbre de Cherednik elliptique aux racines de l'unité. Il sera intéressant d'étudier ces représentations plus précisément, et de comprendre les conséquences de ces résultats. D'autres développements possibles sont évoqués dans le chapitre 5.

Plus généralement, cette thèse est une contribution à la compréhension de la catégorie des représentations de dimension finie de l'algèbre toroïdale quantique aux racines de l'unité. En effet, nous avons obtenu des représentations irréductibles de dimension finie de l'algèbre toroïdale quantique. Une première question, difficile, serait de savoir s'il y en a d'autres, ou si nous avons déterminé ici tous les objets simples de cette catégorie. Comme c'est déjà le cas pour les groupes quantiques et les algèbres affines quantiques aux racines de l'unité [9], les conséquences de tels résultats pourraient être multiples et les applications nombreuses et variées.

Organisation de la thèse

On explique à présent la structure de cette thèse. Comme nous l'avons évoqué, elle est constituée essentiellement de trois parties indépendantes qui peuvent être étudiées séparément :

1. *Extremal loop weight modules and monomial crystals* (article [Man12c] accepté pour publication dans *Pacific Journal of Mathematics*),
2. *Extremal loop weight modules and tensor products for quantum toroïdal algebras* (article [Man13a] prépublié arXiv:1305.3481),
3. *Extremal loop weight modules for* $\mathcal{U}_q(\hat{sl}_\infty)$ (article [Man13b] en préparation).

Les connaissances nécessaires sur les algèbres affines quantiques et les algèbres toroïdales quantiques (notamment les notions de représentations extrémales et de représentations ℓ-extrémales) sont rappelées dans le Chapitre 1. Dans le dernier chapitre de cette thèse, nous présentons un exemple de construction de représentation ℓ-extrémale en type D_4. De plus nous donnons des développements futurs possibles à ces travaux.

9. Nous faisons référence ici aux exemples d'applications des groupes quantiques et des algèbres affines quantiques aux racines de l'unité donnés dans cette introduction.

Chapter 1

Background

We recall the main definitions and general properties about the representation theory of quantum affine algebras and quantum toroidal algebras of type A.

1.1 Cartan matrix

Let $C = (C_{i,j})_{0 \leq i,j \leq n}$ be a Cartan matrix of type $A_n^{(1)}$ $(n \geq 2)$,

$$C = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Remark 1.1.1. *The case $n = 1$ is not studied here and is particular. In this case, the Cartan matrix is*

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

and involves -2. Furthermore, the quantum toroidal algebra $\mathcal{U}_q(sl_2^{tor})$ requires a special definition with different possible choices of the quantized Cartan matrix (see [Her11, Remark 4.1]).

Set $I = \{0, \ldots, n\}$ and $I_0 = \{1, \ldots, n\}$. In particular, $(C_{i,j})_{i,j \in I_0}$ is the Cartan matrix of finite type A_n. In the following, I will be often identified with the set $\mathbb{Z}/(n+1)\mathbb{Z}$. Consider the vector space of dimension $n+2$

$$\mathfrak{h} = \mathbb{Q}h_0 \oplus \mathbb{Q}h_1 \oplus \cdots \oplus \mathbb{Q}h_n \oplus \mathbb{Q}d$$

and the linear functions α_i (the simple roots), Λ_i (the fundamental weights) on $\mathfrak{h}$ given by $(i, j \in I)$

$$\alpha_i(h_j) = C_{j,i}, \quad \alpha_i(d) = \delta_{0,i},$$
$$\Lambda_i(h_j) = \delta_{i,j}, \quad \Lambda_i(d) = 0.$$

Denote by $\Pi = \{\alpha_0, \dots, \alpha_n\} \subset \mathfrak{h}^*$ the set of simple roots and $\Pi^\vee = \{h_0, \dots, h_n\} \subset \mathfrak{h}$ the set of simple coroots. Let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for any } i \in I\}$ be the weight lattice and $P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for any } i \in I\}$ the semigroup of dominant weights. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset P$ (the root lattice) and $Q^+ = \sum_{i\in I} \mathbb{N}\alpha_i \subset Q$. For $\lambda, \mu \in \mathfrak{h}^*$, write $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Set $\mathfrak{h}_0 = \mathbb{Q}h_1 \oplus \dots \oplus \mathbb{Q}h_n$ and $\Pi_0 = \{\alpha_1, \dots, \alpha_n\}$, $\Pi_0^\vee = \{h_1, \dots, h_n\}$. $(\mathfrak{h}_0, \Pi_0, \Pi_0^\vee)$ is a realization of $(C_{i,j})_{i,j\in I_0}$ (see [Kac90]). We define as above the associated weight lattice P_0, its subset P_0^+ of dominant weights, and the root lattice Q_0.

Denote by W the affine Weyl group: it is the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections $s_i \in GL(\mathfrak{h}^*)$ defined by $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ ($i \in I$). The Weyl group of finite type W_0 is the subgroup of W generated by the s_i with $i \in I_0$.

Let $c = h_0 + \dots + h_n$ and $\delta = \alpha_0 + \dots + \alpha_n$. We have

$$\{\omega \in P \mid \omega(h_i) = 0 \text{ for all } i \in I\} = \mathbb{Q}\delta.$$

Put $P_{\mathrm{cl}} = P/\mathbb{Q}\delta$ and denote by $\mathrm{cl} : P \to P_{\mathrm{cl}}$ the canonical projection. Denote by $P^0 = \{\lambda \in P \mid \lambda(c) = 0\}$ the set of level 0 weights.

1.2 Quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$

1.2.1 Definition of $\mathcal{U}_q(\hat{sl}_{n+1})$

We assume in the following that $q = e^t \in \mathbb{C}^*$ ($t \in \mathbb{C}$) is not a root of unity and is fixed. For $l \in \mathbb{Z}, r \geq 0, m \geq m' \geq 0$ we set

$$[l]_q = \frac{q^l - q^{-l}}{q - q^{-1}} \in \mathbb{Z}[q^{\pm 1}],\ [r]_q! = [r]_q[r-1]_q \dots [1]_q,\ \begin{bmatrix} m \\ m' \end{bmatrix}_q = \frac{[m]_q!}{[m-m']_q![m']_q!}.$$

Definition 1.2.1. *The quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$ is the $\mathbb{C}$-algebra with generators k_h ($h \in \mathfrak{h}$), $x_i^\pm$ ($i \in I$) and relations*

$$k_h k_{h'} = k_{h+h'}\ ,\ k_0 = 1,$$

$$k_h x_j^\pm k_{-h} = q^{\pm\alpha_j(h)} x_j^\pm,$$

$$[x_i^+, x_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$(x_i^\pm)^{(2)} x_{i+1}^\pm - x_i^\pm x_{i+1}^\pm x_i^\pm + x_{i+1}^\pm (x_i^\pm)^{(2)} = 0.$$

Here we use the notation $k_i^{\pm 1} = k_{\pm h_i}$ and for all $r \geq 0$ we set $(x_i^\pm)^{(r)} = \frac{(x_i^\pm)^r}{[r]_q!}$. One defines a coproduct on $\mathcal{U}_q(\hat{sl}_{n+1})$ by setting

$$\Delta(k_h) = k_h \otimes k_h,$$

$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i^+ \otimes x_i^+\ ,\ \Delta(x_i^-) = x_i^- \otimes k_i^- + 1 \otimes x_i^-.$$

1.2.2 Subalgebras of $\mathcal{U}_q(\hat{sl}_{n+1})$

Let $\mathcal{U}_q(\hat{sl}_{n+1})'$ be the subalgebra of $\mathcal{U}_q(\hat{sl}_{n+1})$ generated by $x_i^{\pm}$ and k_h $(h \in \sum \mathbb{Q}h_i)$. This has P_{cl} as a weight lattice.

For $J \subset I$ denote by $\mathcal{U}_q(\hat{sl}_{n+1})_J$ the subalgebra of $\mathcal{U}_q(\hat{sl}_{n+1})$ generated by the $x_i^{\pm}, k_{ph_i}$ for $i \in J, p \in \mathbb{Q}$. If $J = I_0$, $\mathcal{U}_q(\hat{sl}_{n+1})_{I_0}$ is the quantum group of finite type associated to the data $(\mathfrak{h}_0, \Pi_0, \Pi_0^\vee)$, also denoted by $\mathcal{U}_q(sl_{n+1})$. In particular, a $\mathcal{U}_q(\hat{sl}_{n+1})$-module has a structure of $\mathcal{U}_q(sl_{n+1})$-module. If $J = \{i\}$ with $i \in I$, $\mathcal{U}_q(\hat{sl}_{n+1})_J$ is isomorphic to $\mathcal{U}_q(sl_2)$ and denoted by $\mathcal{U}_i$. So a $\mathcal{U}_q(\hat{sl}_{n+1})$-module has also a structure of $\mathcal{U}_q(sl_2)$-module.

Let $\mathcal{U}_q(\hat{sl}_{n+1})^+$ (resp. $\mathcal{U}_q(\hat{sl}_{n+1})^-$, $\mathcal{U}_q(\mathfrak{h})$) be the subalgebra of $\mathcal{U}_q(\hat{sl}_{n+1})$ generated by the x_i^+ (resp. the x_i^-, the k_h). We have a triangular decomposition of $\mathcal{U}_q(\hat{sl}_{n+1})$ (see [Lus93]):

Theorem 1.2.2. *We have an isomorphism of vector spaces*

$$\mathcal{U}_q(\hat{sl}_{n+1}) \simeq \mathcal{U}_q(\hat{sl}_{n+1})^- \otimes \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q(\hat{sl}_{n+1})^+.$$

1.3 Representations of $\mathcal{U}_q(\hat{sl}_{n+1})$

1.3.1 Basic properties

For V a representation of $\mathcal{U}_q(\hat{sl}_{n+1})$ and $\nu \in P$, the weight space V_ν of V is

$$V_\nu = \{v \in V | k_h \cdot v = q^{\nu(h)} v, \forall h \in \mathfrak{h}\}.$$

Set $\mathrm{wt}(V) = \{\nu \in P | V_\nu \neq \{0\}\}$.

Definition 1.3.1. *A representation V is said to be in the category $\mathcal{O}$ if*

(i) it admits a weight space decomposition $V = \bigoplus_{\nu \in P} V_\nu$,

(ii) V_ν is finite-dimensional for all ν,

(iii) $\{\nu | V_\nu \neq \{0\}\} \subset \bigcup_{j=1 \cdots N} \{\nu | \nu \leq \lambda_j\}$ for some $\lambda_1, \cdots, \lambda_N \in P$.

For $\lambda \in P$, a representation V is said to be of highest weight λ if there is $v \in V_\lambda$ such that for all $i \in I, x_i^+ \cdot v = 0$ and $\mathcal{U}_q(\hat{sl}_{n+1}) \cdot v = V$. Such a representation is in the category $\mathcal{O}$. Furthermore there is a unique simple highest weight module of highest weight λ.

Definition 1.3.2. *A representation V is said to be integrable if*

(i) it admits a weight space decomposition $V = \bigoplus_{\nu \in P} V_\nu$,

(ii) all the $x_i^{\pm}$ $(i \in I)$ are locally nilpotent.

Remark 1.3.3. *This definition differs from the one given in [Her09]. In fact it is required in addition in [Her09] that the representation V satisfies*

(iii) V_ν is finite-dimensional for any $\nu \in P$,

(iv) $V_{\nu \pm N\alpha_i} = \{0\}$ for all $\nu \in P$, $N >> 0$, $i \in I$.

These conditions are implied by the previous ones for the highest weight modules.

Theorem 1.3.4. *[Lus93] The simple highest weight module of highest weight λ is integrable if and only if λ is dominant. We denote it $V(\lambda)$ $(\lambda \in P^+)$.*

For an integrable representation V of $\mathcal{U}_q(\hat{sl}_{n+1})$ with finite-dimensional weight spaces, one defines the usual character

$$\chi(V) = \sum_{\nu \in P} \dim(V_\nu) e(\nu) \in \prod_{\nu \in P} \mathbb{Z} e(\nu).$$

1.3.2 Finite-dimensional representations of $\mathcal{U}_q(sl_{n+1})$

Similar definitions hold for the quantum group $\mathcal{U}_q(sl_{n+1})$. In this case, the integrable simple highest weight modules are parametrized by P_0^+ and denoted by $V_0(\lambda)$ ($\lambda \in P_0^+$). Further they are finite-dimensional (see [Lus93, Ros91]). Let $\mathcal{C}$ be the category of integrable finite-dimensional representations of $\mathcal{U}_q(sl_{n+1})$ and $\mathcal{R}$ its Grothendieck ring.

Theorem 1.3.5. *[Lus93, Ros91] The category $\mathcal{C}$ is a semi-simple tensor category and the simple objects of $\mathcal{C}$ are the $(V_0(\lambda))_{\lambda \in P_0^+}$. Furthermore χ induces a ring morphism*

$$\chi : \mathcal{R} \to \bigoplus_{\nu \in P_0} \mathbb{Z}e(\nu)$$

where the product at the right hand side is defined by $e(\mu)e(\nu) = e(\mu + \nu)$.

We do not recall here the theory of crystal bases of quantum groups, we just refer to [Kas94, Kas02a, Kas02b]. Let us remind only that for $\lambda \in P^+$, the $\mathcal{U}_q(\hat{sl}_{n+1})$-module $V(\lambda)$ has a crystal basis $\mathcal{B}(\lambda)$. In the same way we denote by $\mathcal{B}_0(\lambda)$ the crystal basis of the $\mathcal{U}_q(sl_{n+1})$-module $V_0(\lambda)$ ($\lambda \in P_0^+$). When we want to distinguish crystals of $\mathcal{U}_q(\hat{sl}_{n+1})$, $\mathcal{U}_q(\hat{sl}_{n+1})_J$ with $J \subset I$ and $\mathcal{U}_q(\hat{sl}_{n+1})'$, we call it respectively a P-crystal or I-crystal, a J-crystal and a P_{cl}-crystal.

1.4 Extremal weight modules

1.4.1 Definition and first results

In this section we recall the definition and some properties of extremal weight modules for the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$ given by Kashiwara [Kas94, Kas02b]. All of these hold for general quantum Kac-Moody algebras and in particular for $\mathcal{U}_q(sl_{n+1})$.

Definition 1.4.1. *For an integrable $\mathcal{U}_q(\hat{sl}_{n+1})$-module V and $\lambda \in P$, a vector $v \in V_\lambda$ is called extremal of weight λ if there are vectors $\{v_w\}_{w \in W}$ such that $v_{Id} = v$ and*

$$x_i^{\pm} \cdot v_w = 0 \textit{ and } (x_i^{\mp})^{(\pm w(\lambda)(h_i))} \cdot v_w = v_{s_i(w)} \textit{ if } \pm w(\lambda)(h_i) \geq 0.$$

Note that if the vector v is extremal of weight λ, then for $w \in W$, v_w is extremal of weight $w(\lambda)$.

Remark 1.4.2. *The definition of extremal vector can be rewritten as follows (see [Kas94]): for an integrable $\mathcal{U}_q(\hat{sl}_{n+1})$-module V, a weight vector v of weight λ is called i-extremal if $x_i^+ \cdot v = 0$ or $x_i^- \cdot v = 0$. In this case we set*

$$S_i(v) = (x_i^-)^{(\lambda(h_i))} \cdot v \textit{ or } S_i(v) = (x_i^+)^{(-\lambda(h_i))} \cdot v \textit{ respectively.}$$

Then a weight vector v is extremal if, for any $l \geq 0$, $S_{i_1} \circ \cdots \circ S_{i_l}(v)$ is i-extremal for any $i, i_1, \cdots, i_l \in I$. We set in that case

$$W \cdot v = \{S_{i_1} \circ \cdots \circ S_{i_l}(v) | \ l \in \mathbb{N}, \ i_1, \cdots, i_l \in I\}.$$

The notion of extremal elements in a crystal $\mathcal{B}$ can be defined in the same way.

Definition 1.4.3. *For $\lambda \in P$, the extremal weight module $V(\lambda)$ of extremal weight λ is the $\mathcal{U}_q(\hat{sl}_{n+1})$-module generated by a vector v_λ with the defining relations that v_λ is extremal of weight λ.*

Example 1.4.4. *If λ is dominant, $V(\lambda)$ is the simple highest weight module of highest weight λ.*

Theorem 1.4.5. *[Kas94] For $\lambda \in P$, the module $V(\lambda)$ is integrable and has a crystal basis $\mathcal{B}(\lambda)$.*

Remark 1.4.6. *To prove this result, the tensor product $V'(\lambda) = V(\lambda_+) \otimes V(\lambda_-)$ of a highest weight module and a lowest weight module is considered in [Kas94], where*

$$\lambda_+ = \sum_{\lambda(h_i) \geq 0} \lambda(h_i)\Lambda_i \text{ and } \lambda_- = \lambda_+ - \lambda \in P_+.$$

Actually, this tensor product is closely related to the extremal weight module $V(\lambda)$ (see [Kas94, Lemma 8.2.1]).

1.4.2 Finite-dimensional quotients

Set $\lambda = \varpi_\ell$, where $1 \leq \ell \leq n$ and ϖ_ℓ is the level 0 fundamental weight $\varpi_\ell = \Lambda_\ell - \Lambda_0$.

Theorem 1.4.7. *[Kas02b] Let $1 \leq \ell \leq n$.*

(i) $V(\varpi_\ell)$ is an irreducible $\mathcal{U}_q(\hat{sl}_{n+1})$-module.

(ii) Any non-zero integrable $\mathcal{U}_q(\hat{sl}_{n+1})$-module generated by an extremal weight vector of weight ϖ_ℓ is isomorphic to $V(\varpi_\ell)$.

The representations $V(\varpi_\ell)$ have a particular importance because they have finite-dimensional quotients over $\mathcal{U}_q(\hat{sl}_{n+1})'$. In fact let w be an element of W such that $w(\varpi_\ell) = \varpi_\ell + \delta$. Such an element exists and is not unique (see [Kas02b]). It defines a $\mathcal{U}_q(\hat{sl}_{n+1})'$-automorphism (also called P_{cl}-automorphism in the following) of the restricted $\mathcal{U}_q(\hat{sl}_{n+1})'$-module $V(\varpi_\ell)$, which sends v to v_w. It is of weight δ, and denoted by z_ℓ. Let us define the $\mathcal{U}_q(\hat{sl}_{n+1})'$-module

$$W(\varpi_\ell) = V(\varpi_\ell)/(z_\ell - 1)V(\varpi_\ell).$$

Then we have

Theorem 1.4.8. *[Kas02b] Let $1 \leq \ell \leq n$.*

(i) $W(\varpi_\ell)$ is a finite-dimensional irreducible $\mathcal{U}_q(\hat{sl}_{n+1})'$-module.

(ii) For any $\mu \in \mathrm{wt}(V(\varpi_\ell))$,

$$W(\varpi_\ell)_{\mathrm{cl}(\mu)} \simeq V(\varpi_\ell)_\mu.$$

(iii) $V(\varpi_\ell)$ is isomorphic to $W(\varpi_\ell)_{\mathrm{aff}}$ as a $\mathcal{U}_q(\hat{sl}_{n+1})$-module.

Here M_{aff} is the affinization of an integrable $\mathcal{U}_q(\hat{sl}_{n+1})'$-module M: this is the $\mathcal{U}_q(\hat{sl}_{n+1})$-module with a weight space decomposition $M_{\mathrm{aff}} = \bigoplus_{\nu \in P} (M_{\mathrm{aff}})_\nu$ defined by

$$(M_{\mathrm{aff}})_\nu = M_{\mathrm{cl}(\nu)}$$

and with the obvious action of $x_i^\pm$. Note also that we have an isomorphism of $\mathcal{U}_q(\hat{sl}_{n+1})'$-modules

$$M_{\mathrm{aff}} \simeq \mathbb{C}[z, z^{-1}] \otimes M$$

where $x_i^\pm$ act on the right hand side by $z^{\pm\delta_{i,0}} x_i^\pm$. In the same way one defines the affinization $\mathcal{B}_{\mathrm{aff}}$ of a P_{cl}-crystal $\mathcal{B}$. For an integrable $\mathcal{U}_q(\hat{sl}_{n+1})'$-module M with associated P_{cl}-crystal $\mathcal{B}$, its affinization M_{aff} has a P-crystal $\mathcal{B}_{\mathrm{aff}}$.

This last result implies in particular the existence of crystal bases for finite-dimensional fundamental representations of $\mathcal{U}_q(\hat{sl}_{n+1})'$ (see Section 1.6.6).

1.5 Quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$

In this section, we recall the definition and the main properties of the quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$ (without central charge).

1.5.1 Definition

From the affine Lie algebra $\hat{sl}_{n+1}$, one can associate its Lie algebra of currents $\mathcal{L}(\hat{sl}_{n+1}) = \mathbb{C}[t, t^{-1}] \otimes \hat{sl}_{n+1}$, also called the toroidal algebra of type A. Here one gives a quantum version of $\mathcal{L}(\hat{sl}_{n+1})$. It is written in terms of Drinfeld generators.

Definition 1.5.1. *[GKV95] The quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$ is the $\mathbb{C}$-algebra with generators $x_{i,r}^{\pm}$ $(i \in I, r \in \mathbb{Z})$, k_h $(h \in \mathfrak{h})$, $h_{i,m}$ $(i \in I, m \in \mathbb{Z} - \{0\})$ and the following relations $(i, j \in I, r, r', r_1, r_2 \in \mathbb{Z}, m \in \mathbb{Z} - \{0\})$*

$$\begin{gathered}
k_h k_{h'} = k_{h+h'} \ , \ k_0 = 1 \ , \ [k_h, h_{j,m}] = 0 \ , \ [h_{i,m}, h_{j,m'}] = 0, \\
k_h x_{j,r}^{\pm} k_{-h} = q^{\pm \alpha_j(h)} x_{j,r}^{\pm}, \\
[h_{i,m}, x_{j,r}^{\pm}] = \pm \frac{1}{m} [m C_{i,j}]_q x_{j,m+r}^{\pm}, \\
[x_{i,r}^{+}, x_{j,r'}^{-}] = \delta_{ij} \frac{\phi_{i,r+r'}^{+} - \phi_{i,r+r'}^{-}}{q - q^{-1}}, \\
x_{i,r+1}^{\pm} x_{j,r'}^{\pm} - q^{\pm C_{ij}} x_{j,r'}^{\pm} x_{i,r+1}^{\pm} = q^{\pm C_{ij}} x_{i,r}^{\pm} x_{j,r'+1}^{\pm} - x_{j,r'+1}^{\pm} x_{i,r}^{\pm},
\end{gathered} \tag{1.1}$$

$$\begin{gathered}
x_{i,r_1}^{\pm} x_{i,r_2}^{\pm} x_{i\pm1,r'}^{\pm} - (q + q^{-1}) x_{i,r_1}^{\pm} x_{i\pm1,r'}^{\pm} x_{i,r_2}^{\pm} + x_{i\pm1,r'}^{\pm} x_{i,r_1}^{\pm} x_{i,r_2}^{\pm} = \\
-x_{i,r_2}^{\pm} x_{i,r_1}^{\pm} x_{i\pm1,r'}^{\pm} + (q + q^{-1}) x_{i,r_2}^{\pm} x_{i\pm1,r'}^{\pm} x_{i,r_1}^{\pm} - x_{i\pm1,r'}^{\pm} x_{i,r_2}^{\pm} x_{i,r_1}^{\pm},
\end{gathered}$$

and $[x_{i,r_1}^{\pm}, x_{j,r_2}^{\pm}] = 0$ if $i \neq j, j \pm 1$. Here for all $i \in I$ and $m \in \mathbb{Z}$, $\phi_{i,m}^{\pm} \in \mathcal{U}_q(sl_{n+1}^{tor})$ is determined by the formal power series in $\mathcal{U}_q(sl_{n+1}^{tor})[[z]]$ (resp. in $\mathcal{U}_q(sl_{n+1}^{tor})[[z^{-1}]]$)

$$\phi_i^{\pm}(z) = \sum_{m \geq 0} \phi_{i,\pm m}^{\pm} z^{\pm m} = k_i^{\pm 1} \exp \left(\pm (q - q^{-1}) \sum_{m' \geq 1} h_{i,\pm m'} z^{\pm m'} \right)$$

and $\phi_{i,m}^{+} = 0$ for $m < 0$, $\phi_{i,m}^{-} = 0$ for $m > 0$.

Remark 1.5.2. *The generators $x_{i,r}^{\pm}$ and $h_{i,m}$ are quantum versions of $t^r \otimes x_i^{\pm}$ and $t^m \otimes k_i$ $(i \in I, r \in \mathbb{Z}, m \in \mathbb{Z} - \{0\})$.*

Remark 1.5.3. *The affine Lie algebra $\hat{sl}_{n+1}$ is the Kac-Moody algebra associated to the Cartan matrix C of type $A_n^{(1)}$ [Kac90]. A crucial point is that $\hat{sl}_{n+1}$ has another realization, as a central extension with one-dimensional centre of the algebra of currents $\mathcal{L}(sl_{n+1}) = \mathbb{C}[t, t^{-1}] \otimes sl_{n+1}$. This phenomenon recovers in the quantum case.*

Theorem 1.5.4. *[Bec94, Dri88] The quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})'$ has another realization in terms of Drinfeld generators: this is the $\mathbb{C}$-algebra with generators $x_{i,r}^{\pm}$ $(i \in I_0, r \in \mathbb{Z})$, k_h $(h \in \mathfrak{h}_0)$, $h_{i,m}$ $(i \in I_0, m \in \mathbb{Z} - \{0\})$ and the same relations as in Definition 1.5.1.*

1.5.2 Definition in terms of currents

For $i \in I$, consider the formal power series (also called currents)

$$\delta(z) = \sum_{s\in\mathbb{Z}} z^s,\ x_i^{\pm}(z) = \sum_{r\in\mathbb{Z}} x_{i,r}^{\pm} z^r.$$

The defining relations of $\mathcal{U}_q(sl_{n+1}^{tor})$ can be rewritten with currents ($i, j \in I, h, h' \in \mathfrak{h}$):

$$k_h k_{h'} = k_{h+h'}\ ,\ k_0 = 1,$$

$$\phi_i^{\pm}(z)\phi_j^{\pm}(w) = \phi_j^{\pm}(w)\phi_i^{\pm}(z)\ ,\ \phi_i^{-}(z)\phi_j^{+}(w) = \phi_j^{+}(w)\phi_i^{-}(z),$$

$$(w - q^{\pm C_{ij}}z)\phi_i^{+}(z)x_j^{\pm}(w) = (q^{\pm C_{ij}}w - z)x_j^{\pm}(w)\phi_i^{+}(z), \tag{1.2}$$

$$(w - q^{\pm C_{ij}}z)\phi_i^{-}(z)x_j^{\pm}(w) = (q^{\pm C_{ij}}w - z)x_j^{\pm}(w)\phi_i^{-}(z), \tag{1.3}$$

$$[x_i^{+}(z), x_j^{-}(w)] = \frac{\delta_{i,j}}{q - q^{-1}}(\delta(\frac{w}{z})\phi_i^{+}(w) - \delta(\frac{z}{w})\phi_i^{-}(z)),$$

$$(z - q^{\pm C_{ij}}w)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\pm C_{ij}}z - w)x_j^{\pm}(w)x_i^{\pm}(z), \tag{1.4}$$

$$\begin{gathered} x_i^{\pm}(z_1)x_i^{\pm}(z_2)x_{i\pm1}^{\pm}(w) - (q + q^{-1})x_i^{\pm}(z_1)x_{i\pm1}^{\pm}(w)x_i^{\pm}(z_2) \\ + x_{i\pm1}^{\pm}(w)x_i^{\pm}(z_1)x_i^{\pm}(z_2) + (z_1 \leftrightarrow z_2) = 0, \end{gathered}$$

and $[x_i^{\pm}(z), x_j^{\pm}(w)] = 0$ for $i \neq j, j \pm 1$.

This definition with currents is used in the whole Chapter 3.

Remark 1.5.5. *An additional parameter $d \in \mathbb{C}^*$ is added in [GKV95]. To do that, consider a matrix $M = (m_{i,j})_{i,j\in I}$ given by $m_{i-1,i} = -1, m_{i,i-1} = 1$ for all $i \in I$ and $m_{i,j} = 0$ if $j \in I, j \neq i, i \pm 1$. Then $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ is the algebra with the same generators and relations as $\mathcal{U}_q(sl_{n+1}^{tor})$ except the following d-deformed relations instead of (1.2), (1.3), (1.4):*

$$(w - d^{m_{ij}}q^{\pm C_{ij}}z)\phi_i^{+}(z)x_j^{\pm}(w) = (q^{\pm C_{ij}}w - d^{m_{ij}}z)x_j^{\pm}(w)\phi_i^{+}(z),$$

$$(w - d^{m_{ij}}q^{\pm C_{ij}}z)\phi_i^{-}(z)x_j^{\pm}(w) = (q^{\pm C_{ij}}w - d^{m_{ij}}z)x_j^{\pm}(w)\phi_i^{-}(z),$$

$$(d^{m_{ij}}z - q^{\pm C_{ij}}w)x_i^{\pm}(z)x_j^{\pm}(w) = (d^{m_{ij}}q^{\pm C_{ij}}z - w)x_j^{\pm}(w)x_i^{\pm}(z).$$

This algebra and their representations are also studied in [FJMM13].

1.5.3 Horizontal and vertical quantum affine subalgebras

There is an algebra morphism $\mathcal{U}_q(\hat{sl}_{n+1}) \to \mathcal{U}_q(sl_{n+1}^{tor})$ defined by

$$k_h \mapsto k_h,\ x_i^{\pm} \mapsto x_{i,0}^{\pm} \text{ with } h \in \mathfrak{h}, i \in I.$$

Its image is called the *horizontal quantum affine subalgebra* of $\mathcal{U}_q(sl_{n+1}^{tor})$ and is denoted by $\mathcal{U}_q^h(sl_{n+1}^{tor})$. In particular, a $\mathcal{U}_q(sl_{n+1}^{tor})$-module V has also a structure of $\mathcal{U}_q(\hat{sl}_{n+1})$-module. We denote by $\mathrm{Res}(V)$ the restricted $\mathcal{U}_q(\hat{sl}_{n+1})$-module obtained from V.

The quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})'$ recovers in another way as subalgebra of $\mathcal{U}_q(sl_{n+1}^{tor})$: by Theorem 1.5.4 it is isomorphic to the subalgebra $\mathcal{U}_q^v(sl_{n+1}^{tor})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$ generated by the $x_{i,r}^{\pm}$, k_h, $h_{i,m}$ $(i \in I_0, r \in \mathbb{Z}, h \in \mathfrak{h}_0, m \in \mathbb{Z} - \{0\})$. $\mathcal{U}_q^v(sl_{n+1}^{tor})$ is called the *vertical quantum affine subalgebra* of $\mathcal{U}_q(sl_{n+1}^{tor})$.

For all $j \in I$, set $I_j = I - \{j\}$ and define the subalgebra $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$ generated by the $x_{i,r}^{\pm}$, k_h, $h_{i,m}$ $(i \in I_j, r \in \mathbb{Z}, h \in \bigoplus_{i \in I_j} \mathbb{Q}h_i, m \in \mathbb{Z} - \{0\})$. In particular $\mathcal{U}_q^{v,0}(sl_{n+1}^{tor})$ is the vertical quantum affine subalgebra $\mathcal{U}_q^v(sl_{n+1}^{tor})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$. All the $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ for various $j \in I$ are isomorphic: in fact let θ be the automorphism of the Dynkin diagram of type $A_n^{(1)}$ corresponding to the rotation such that $\theta(k) = k+1$, where I is identified to the set $\mathbb{Z}/(n+1)\mathbb{Z}$. It defines an automorphism $\theta_{\mathfrak{h}}$ of $\mathfrak{h}$ by sending h_i, d to $h_{\theta(i)}, d$ $(i \in I)$. For all $j \in J$, let $\theta^{(j)}$ be the automorphism of $\mathcal{U}_q(sl_{n+1}^{tor})$ which sends $x_{i,r}^{\pm}$, k_h, $h_{i,m}$ to $x_{\theta^j(i),r}^{\pm}$, $k_{\theta_{\mathfrak{h}}^j(h)}$, $h_{\theta^j(i),m}$ respectively (where $i \in I$, $h \in \mathfrak{h}$, $r \in \mathbb{Z}$, $m \in \mathbb{Z} - \{0\}$). It gives by restriction an isomorphism of algebras between $\mathcal{U}_q^v(sl_{n+1}^{tor})$ and $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$, still denoted by $\theta^{(j)}$ in the following. If V is a $\mathcal{U}_q(\hat{sl}_{n+1})'$-module, we denote by $V^{(j)}$ the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module.

Hence the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$ recovers in two ways as subalgebra of $\mathcal{U}_q(sl_{n+1}^{tor})$. This crucial point involves in the definition of extremal loop weight modules and is essential in the whole thesis.

1.5.4 Basic properties

For $b \in \mathbb{C}^*$, let t_b be the map sending the series $x_i^{\pm}(z), \phi_i^{\pm}(z)$ to $x_i^{\pm}(bz), \phi_i^{\pm}(bz)$, respectively $(i \in I)$. Hence defined, t_b is an automorphism of $\mathcal{U}_q(sl_{n+1}^{tor})$.

For $i \in I$, the subalgebra $\hat{\mathcal{U}}_i$ generated by the $x_{i,r}^{\pm}, h_{i,m}, k_{ph_i}$ $(r \in \mathbb{Z}$, $m \in \mathbb{Z} - \{0\}$, $p \in \mathbb{Q})$ is isomorphic to $\mathcal{U}_q(\hat{sl}_2)'$.

We have a triangular decomposition of $\mathcal{U}_q(sl_{n+1}^{tor})$.

Theorem 1.5.6. *[Mik00, Nak01] We have an isomorphism of vector spaces*

$$\mathcal{U}_q(sl_{n+1}^{tor}) \simeq \mathcal{U}_q(sl_{n+1}^{tor})^- \otimes \mathcal{U}_q(\hat{\mathfrak{h}}) \otimes \mathcal{U}_q(sl_{n+1}^{tor})^+,$$

where $\mathcal{U}_q(sl_{n+1}^{tor})^{\pm}$ (resp. $\mathcal{U}_q(\hat{\mathfrak{h}})$) is generated by the $x_{i,r}^{\pm}$ (resp. the k_h, the $h_{i,m}$).

1.6 Representations of $\mathcal{U}_q(sl_{n+1}^{tor})$

1.6.1 Representations of ℓ-highest weight

Definition 1.6.1. *A representation V of $\mathcal{U}_q(sl_{n+1}^{tor})$ is said to be integrable (resp. in the category $\mathcal{O}$) if $\mathrm{Res}(V)$ is integrable (resp. in the category $\mathcal{O}$) as a $\mathcal{U}_q(\hat{sl}_{n+1})$-module.*

The definition of highest weight representations for $\mathcal{U}_q(sl_{n+1}^{tor})$ (and for general quantum affinizations) does not make sense with Drinfeld generators. However an analogous notion of ℓ-highest weight modules exists.

Definition 1.6.2. *A representation V of $\mathcal{U}_q(sl_{n+1}^{tor})$ is said to be of ℓ-highest weight if there is $v \in V$ such that*

(i) $V = \mathcal{U}_q(sl_{n+1}^{tor})^- \cdot v$,

(ii) there exists $\gamma \in \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}), \mathbb{C})$ (called ℓ-highest weight) such that

$$\phi_{i,\pm m}^{\pm} \cdot v = \gamma(\phi_{i,\pm m}^{\pm})v \text{ for all } i \in I \text{ and } m \geq 0,$$

(iii) for any $i \in I, r \in \mathbb{Z}$, $x_{i,r}^{+} \cdot v = 0$.

Such a representation is in the category $\mathcal{O}$. For $\gamma \in \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}), \mathbb{C})$ an algebra morphism, by Theorem 1.5.6 we have a corresponding Verma module $M(\gamma)$ and a simple representation $V(\gamma)$ which are ℓ-highest weight.

Theorem 1.6.3. *[Mik00, Nak01] The simple representations $V(\gamma)$ of $\mathcal{U}_q(sl_{n+1}^{tor})$ are integrable if there is $(\lambda, (P_i)_{i\in I}) \in P^+ \times (1 + u\mathbb{C}[u])^I$ satisfying $\gamma(k_h) = q^{\lambda(h)}$ and for $i \in I$ the relation in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$)*

$$\gamma(\phi_i^{\pm}(z)) = q^{deg(P_i)} \frac{P_i(zq^{-1})}{P_i(zq)}.$$

The polynomials P_i are called Drinfeld polynomials and the representation $V(\gamma)$ is then denoted by $V(\lambda, (P_i)_{i\in I})$. It is infinite-dimensional. Such a representation is also integrable in the sense of [Her09], i.e. $V(\lambda, (P_i)_{i\in I})$ satisfies conditions (iii) and (iv) of Remark 1.3.3.

The Kirillov-Reshetikhin module associated to $k \geq 0$, $a \in \mathbb{C}^*$ and $0 \leq \ell \leq n$, is the simple integrable infinite-dimensional representation of weight $k\Lambda_\ell$ with the n–tuple

$$P_i(u) = \begin{cases} (1 - ua)(1 - uaq^2) \cdots (1 - uaq^{2(k-1)}) \text{ for } i = \ell, \\ 1 \text{ for } i \neq \ell. \end{cases}$$

If $k = 1$, it is also called fundamental module.

1.6.2 Morphism of q-character

Consider an integrable representation V of $\mathcal{U}_q(sl_{n+1}^{tor})$. As the subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ is commutative, we have a decomposition of the weight spaces V_ν in simultaneous generalized eigenspaces

$$V_\nu = \bigoplus_{(\nu,\gamma)\in P\times \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}),\mathbb{C})} V_{(\nu,\gamma)},$$

where $V_{(\nu,\gamma)} = \{x \in V / \exists p \in \mathbb{N}, \forall i \in I, \forall m \geq 0, (\phi_{i,\pm m}^{\pm} - \gamma(\phi_{i,\pm m}^{\pm}))^p \cdot x = 0\}$. If $V_{(\nu,\gamma)} \neq \{0\}$, (ν, γ) is called an ℓ-weight of V.

Definition 1.6.4. *A representation V is said to be ℓ-integrable if*

(i) it admits an ℓ-weight space decomposition $V = \bigoplus V_{(\nu,\gamma)}$,

(ii) for all $v \in V$ and $i \in I$, $\hat{\mathcal{U}}_i \cdot v$ is finite-dimensional.

All the simple ℓ-highest weight representations $V(\lambda, (P_i)_{i\in I})$ are ℓ-integrable. An ℓ-integrable representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ is integrable.

Definition 1.6.5. *A $\mathcal{U}_q(sl_{n+1}^{tor})$-module V is weighted if the Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ acts on V by diagonalizable operators. The module V is thin if it is weighted and the joint spectrum is simple.*

The terminology is different in [FFJ+11a, FFJ+11b, FJMM12, FJMM13]: a thin module is called tame.

Definition 1.6.6. *[FR99, Her05, Nak01] The q–character of an integrable representation V of $\mathcal{U}_q(sl_{n+1}^{tor})$ with finite-dimensional ℓ-weight spaces is defined by the formal sum*

$$\chi_q(V) = \sum_{(\nu,\gamma)\in P\times \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}),\mathbb{C})} \dim(V_{(\nu,\gamma)}) e(\nu, \gamma).$$

Furthermore if the weight spaces of V are finite-dimensional we have

$$\chi(\mathrm{Res}(V)) = \beta(\chi_q(V)),$$

where $\mathrm{Res}(V)$ still denotes the restricted $\mathcal{U}_q(\hat{sl}_{n+1})$-module obtained from V and

$$\beta : \prod_{(\nu,\gamma)\in P\times \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}),\mathbb{C})} \mathbb{Z}e(\nu,\gamma) \to \prod_{\nu\in P} \mathbb{Z}e(\nu)$$

is $\mathbb{Z}$-linear such that $\beta(e(\nu,\gamma)) = e(\nu)$ for all $(\nu,\gamma) \in P \times \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}),\mathbb{C})$.

Proposition 1.6.7. *[FR99, Her05, Nak01] Let V be an integrable representation of $\mathcal{U}_q(sl^{tor}_{n+1})$. An ℓ-weight $(\nu,\gamma) \in P \times \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}),\mathbb{C})$ of V satisfies the property*

(i) there exist polynomials $Q_i(z), R_i(z) \in \mathbb{C}[z]$ $(i \in I)$ of constant term 1 such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$):

$$\sum_{m\geq 0} \gamma(\phi^{\pm}_{i,\pm m}) z^{\pm m} = q^{\deg(Q_i)-\deg(R_i)} \frac{Q_i(zq^{-1})R_i(zq)}{Q_i(zq)R_i(zq^{-1})}. \tag{1.5}$$

Further if V is in the category $\mathcal{O}$, then

(ii) there exist $\omega \in P^+$, $\alpha \in Q^+$ satisfying $\nu = \omega - \alpha$.

As in [FJMM13], we set

$$\psi(z) = \frac{q - q^{-1}z}{1-z}.$$

Then (1.5) can be rewritten as follows: for all $i \in I$, there exist $k, l \in \mathbb{N}$ and $a_{1,i}, \cdots, a_{k,i}$, $b_{1,i}, \cdots, b_{l,i} \in \mathbb{C}^*$ such that $\nu(h_i) = k - l$ and in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$)

$$\begin{aligned}\sum_{m\geq 0} \gamma(\phi^{\pm}_{i,\pm m}) z^{\pm m} &= \prod_{1\leq u\leq k} \psi(a_{u,i}qz) \prod_{1\leq v\leq l} \psi(b_{v,i}qz)^{-1} \\ &= \prod_{1\leq u\leq k} \psi(a_{u,i}qz) \prod_{1\leq v\leq l} \psi(b^{-1}_{v,i}qz^{-1}).\end{aligned}$$

1.6.3 Correspondence between ℓ-weights and monomials

Consider formal variables $Y^{\pm 1}_{i,a}$, e^{ν} $(i \in I,\ a \in \mathbb{C}^*,\ \nu \in P)$ and let A be the group of monomials of the form $m = e^{\omega(m)} \prod_{i\in I, a\in\mathbb{C}^*} Y^{u_{i,a}(m)}_{i,a}$ where $u_{i,a}(m) \in \mathbb{Z}$, $\omega(m) \in P$ are such that

$$\sum_{a\in\mathbb{C}^*} u_{i,a}(m) = \omega(m)(h_i).$$

For example, $Y^{\pm 1}_{i,a} e^{\pm\Lambda_i} \in A$ and $A_{i,a} = e^{\alpha_i} Y_{i,aq^{l-1}} Y_{i,aq^{l+1}} Y^{-1}_{i-1,aq^l} Y^{-1}_{i+1,aq^l} \in A$. A monomial m is said to be J-dominant $(J \subset I)$ if for all $j \in J$ and $a \in \mathbb{C}^*$ we have $u_{j,a}(m) \geq 0$. An I-dominant monomial is said to be dominant.

Remark 1.6.8. *Let us fix a monomial $m \in A$ and consider monomials m' which are products of m with various $A^{\pm 1}_{i,a}$'s $(i \in I, a \in \mathbb{C}^*)$. By [HN06, Remark 2.1], $\omega(m')$ is uniquely determined by $\omega(m)$ and $u_{i,l}(m')$. So in the following when we are in this situation, the term $e^{\omega(m')}$ will be safely omitted.*

Let V be an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module. For (ν,γ) an ℓ-weight of V, one defines the monomial $m_{(\nu,\gamma)} = e^{\nu}\prod_{i\in I}\prod_{a_i\in\mathbb{C}^*} Y_{i,a_i}\prod_{b_i\in\mathbb{C}^*} Y_{i,b_i}^{-1}$ where

$$\sum_{m\geq 0}\gamma(\phi^{\pm}_{i,\pm m})z^{\pm m} = \prod_{a_i}\psi(a_iqz)\prod_{b_i}\psi(b_iqz)^{-1}.$$

We denote $V_{(\nu,\gamma)} = V_{m_{(\nu,\gamma)}}$. We rewrite the q–character of an integrable representation V with finite-dimensional ℓ-weight spaces by the formal sum

$$\chi_q(V) = \sum_m \dim(V_m)m \in \mathbb{Z}[[e^{\nu}, Y_{i,a}^{\pm 1}]]_{\nu\in P, i\in I, a\in\mathbb{C}^*}.$$

Let us denote by $\mathcal{M}(V)$ the set of monomials occurring in $\chi_q(V)$.

By this correspondence between ℓ-weights and monomials due to Frenkel-Reshetikhin [FR99], the I-tuple of Drinfeld polynomials are identified with the dominant monomials. In particular for a dominant monomial m, one denotes by $V(m)$ the simple module of ℓ-highest weight m. For example $V(e^{k\Lambda_\ell}Y_{\ell,a}Y_{\ell,aq^2}\dots Y_{\ell,aq^{2(k-1)}})$ is the Kirillov-Reshetikhin module associated to $k\geq 0$, $a\in\mathbb{C}^*$ and $\ell\in I$, and $V(e^{\Lambda_\ell}Y_{\ell,a})$ is the fundamental module associated to $a\in\mathbb{C}^*$ and $\ell\in I$.

Let $A_{\mathbb{Z}}$ be the subgroup of A consisting of monomials of the form

$$m = e^{\omega(m)}\prod_{i\in I, l\in\mathbb{Z}} Y_{i,q^l}^{u_{i,q^l}(m)}.$$

The following property will be useful in Chapter 2.

Proposition 1.6.9. *[FR99, Nak01, Her05] Let V be an integrable representation of $\mathcal{U}_q(sl_{n+1}^{tor})$. Assume that V has a finite composition series*

$$L_0 = \{0\}\subset L_1\subset L_2\subset\cdots\subset L_k = V$$

such that $L_{j+1}/L_j \simeq V(m)$ with $m\in A_{\mathbb{Z}}$ dominant. Then all its ℓ-weights $m'\in\mathcal{M}(V)$ belong to $A_{\mathbb{Z}}$.

For a Kirillov-Reshetikhin module $V(e^{k\Lambda_\ell}Y_{\ell,a}Y_{\ell,aq^2}\dots Y_{\ell,aq^{2(k-1)}})$, one reduces to the case where the defining parameter a is in $q^{\mathbb{Z}}$ by twisting the action by an automorphism t_b of $\mathcal{U}_q(sl_{n+1}^{tor})$ for some $b\in\mathbb{C}^*$.

1.6.4 Finite-dimensional representations of $\mathcal{U}_q(\hat{sl}_{n+1})'$

Similar results hold for the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})'$ due to Chari-Pressley [CP94]. One uses here the realization of $\mathcal{U}_q(\hat{sl}_{n+1})'$ in terms of Drinfeld generators. In this situation, the simple integrable ℓ-highest [1] weight representations are finite-dimensional and denoted $V_0((P_i)_{i\in I_0})$ in the following. Note that the weights $\lambda\in P_0$ can be omitted here because they are completely determined by the Drinfeld polynomials $(P_i)_{i\in I_0}$,

$$\lambda = \deg(P_1)\Lambda_1 + \cdots + \deg(P_n)\Lambda_n.$$

In the same way if V is a Kirillov-Reshetikhin module of $\mathcal{U}_q(\hat{sl}_{n+1})'$, its ℓ-weights can be only considered as elements of $\mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}_0),\mathbb{C})$ (where $\mathcal{U}_q(\hat{\mathfrak{h}}_0)$ is the subalgebra of

1. Note that the presence of the letter ℓ is crutial here. In fact as the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})'$ has two realizations in terms of Chevalley generators or of Drinfeld generators, there is a notion of highest weight representations and a notion of ℓ-highest weight representations for this algebra.

$\mathcal{U}_q(\hat{sl}_{n+1})'$ generated by k_h ($h \in \mathfrak{h}_0$) and $h_{i,m}$ ($i \in I_0, m \in \mathbb{Z} - \{0\}$)). They still satisfy relations (1.5).

One does not have expression of q–character of a representation in general. But explicit formulas exist for the fundamental modules and the Kirillov-Reshetikhin modules over $\mathcal{U}_q(\hat{sl}_{n+1})'$ and $\mathcal{U}_q(sl^{tor}_{n+1})$, given in terms of tableaux [Nak03, Her11].

Let us recall that the Kirillov-Reshetikhin modules $V_0(Y_{\ell,a}Y_{\ell,aq^2}\dots Y_{\ell,aq^{2(k-1)}})$ ($k \geq 0, a \in \mathbb{C}^*, \ell \in I_0$) over $\mathcal{U}_q(\hat{sl}_{n+1})'$ can be obtained from the $\mathcal{U}_q(sl_{n+1})$-modules $V_0(k\Lambda_\ell)$ as follows: there exist evaluation morphisms $\mathrm{ev}_a : \mathcal{U}_q(\hat{sl}_{n+1})' \to \mathcal{U}_q(sl_{n+1})$ ($a \in \mathbb{C}^*$), which send $x_{i,0}, k_h$ on x_i, k_h respectively [2] ($i \in I_0, h \in \mathfrak{h}_0$). Then the $\mathcal{U}_q(\hat{sl}_{n+1})'$-module $V_0(Y_{\ell,a}Y_{\ell,aq^2}\dots Y_{\ell,aq^{2(k-1)}})$ can be obtained by pulling back the action of $\mathcal{U}_q(sl_{n+1})$ on $V_0(k\Lambda_\ell)$ by ev_a for some $a \in \mathbb{C}^*$. In particular, $V_0(Y_{\ell,a}Y_{\ell,aq^2}\dots Y_{\ell,aq^{2(k-1)}})$ is irreducible as a $\mathcal{U}_q(sl_{n+1})$-module, isomorphic to $V_0(k\Lambda_\ell)$.

1.6.5 The subcategory $\mathcal{C}_{l,\mathbb{Z}}$ of $\mathcal{C}_l$

In Chapter 2, we have to deal with $\mathcal{U}_q(\hat{sl}_{n+1})'$-modules with involving spectral parameters in $q^{\mathbb{Z}}$. We explain in this section who to come down to this.

Let V be a Kirillov-Reshetikhin module of $\mathcal{U}_q(\hat{sl}_{n+1})'$. Then by twisting the action on V by an automorphism t_b of $\mathcal{U}_q(\hat{sl}_{n+1})'$ for some $b \in \mathbb{C}^*$, it can be parametrized as above by Laurent monomials in $\mathbb{Z}[Y^{\pm 1}_{i,q^l}]_{i\in I_0, l\in\mathbb{Z}}$. The weight $\omega(m) \in P_0$ of a monomial $m \in \mathbb{Z}[Y^{\pm 1}_{i,q^l}]_{i\in I_0, l\in\mathbb{Z}}$ can be omitted here because it is completely determined by the $u_{i,q^l}(m)$ ($i \in I_0, l \in \mathbb{Z}$).

Let $\mathcal{C}_l$ be the category of finite-dimensional $\mathcal{U}_q(\hat{sl}_{n+1})'$-modules (of type 1) and $\mathcal{R}_l$ its Grothendieck ring. Recall that $\mathcal{C}_l$ is an abelian monoidal category, not semi-simple, with the $V_0((P_i)_{i\in I_0})$ as simple objects and $\mathcal{R}_l$ is the polynomial ring over $\mathbb{Z}$ in the classes $[V_0((1-\delta_{\ell,i}au)_{i\in I_0})]$ ($\ell \in I_0, a \in \mathbb{C}^*$) (see [CP94, FR99]). As in [HL10], we consider $\mathcal{C}_{l,\mathbb{Z}}$ the full subcategory of $\mathcal{C}_l$ whose objects V satisfy

> for every composition factor S of V, the roots of the Drinfeld polynomials $(P_i(u))_{i\in I_0}$ belong to $q^{\mathbb{Z}}$.

This is also an abelian monoidal category, not semi-simple and the Grothendieck ring $\mathcal{R}_{l,\mathbb{Z}}$ of $\mathcal{C}_{l,\mathbb{Z}}$ is the subring of $\mathcal{R}_l$ generated by the classes $[V_0(Y_{\ell,q^s})]$ with $\ell \in I_0, s \in \mathbb{Z}$ (see [FM01]).

Theorem 1.6.10. *[FR99] χ_q induces a ring morphism $\chi_q : \mathcal{R}_{l,\mathbb{Z}} \to \mathbb{Z}[Y^{\pm 1}_{i,q^l}]_{i\in I_0, l\in\mathbb{Z}}$, satisfying the commutative diagram*

$$\begin{tikzcd} \mathcal{R}_{l,\mathbb{Z}} \arrow[r, "\chi_q"] \arrow[d, "\mathrm{Res}"'] & \mathbb{Z}[Y^{\pm 1}_{i,q^l}]_{i\in I_0, l\in\mathbb{Z}} \arrow[d, "\beta"] \\ \mathcal{R} \arrow[r, "\chi"] & \bigoplus_{\nu\in P_0} \mathbb{Z}e(\nu) \end{tikzcd}$$

where the ring morphism Res *is the restriction and β is defined by $\beta(m) = e(\omega(m))$.*

1.6.6 Link with the $\mathcal{U}_q(\hat{sl}_{n+1})'$-modules $W(\varpi_\ell)$

We have defined irreducible finite-dimensional $\mathcal{U}_q(\hat{sl}_{n+1})'$-modules $W(\varpi_\ell)$ ($\ell \in I_0$) in Section 1.4. One can determine them in terms of the Drinfeld realization. For that we need the following additional result.

2. ev_a is the quantum analogue of the evaluation morphism $\mathcal{L}(sl_{n+1}) \to sl_{n+1}$ which sends $t^r \otimes x$ to $a^r \cdot x$ ($r \in \mathbb{Z}, x \in sl_{n+1}$).

Lemma 1.6.11. *[Nak04] Let v be a vector of an integrable $\mathcal{U}_q(\hat{sl}_{n+1})'$-module of weight $\lambda \in \mathrm{cl}(P^0)$ such that for all $i \in I_0$, $\lambda(h_i) \geq 0$. Then the following conditions are equivalent*

(i) v is an extremal vector,

(ii) $x_{i,r}^+ \cdot v = 0$ for all $i \in I_0, r \in \mathbb{Z}$.

As a direct consequence of these results, $W(\varpi_\ell)$ is isomorphic to a fundamental representation $V_0((1 - \delta_{\ell,i} au)_{i \in I_0})$ for a special choice of the spectral parameter $a \in \mathbb{C}^*$ (see [Nak04, Remark 3.3] for an expression of it). In particular for this spectral parameter, one deduces that $V_0((1 - \delta_{\ell,i} au)_{i \in I_0})$ has a crystal basis.

1.7 Extremal loop weight modules

1.7.1 Definition

We give the definition of extremal loop weight modules for $\mathcal{U}_q(sl_{n+1}^{tor})$. The main motivation is the construction of finite-dimensional representations of the quantum toroidal algebra as in the theory of Kashiwara, but at roots of unity in that case.

Definition 1.7.1. *[Her09] An extremal loop weight module of $\mathcal{U}_q(sl_{n+1}^{tor})$ of ℓ-weight m is an integrable representation V such that there is $v \in V_m$ satisfying*

(i) $\mathcal{U}_q(sl_{n+1}^{tor}) \cdot v = V$,

(ii) v is extremal for $\mathcal{U}_q^h(sl_{n+1}^{tor})$,

(iii) $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot w$ is finite-dimensional for all $w \in V$ and $j \in I$.

An example of such a representation which is neither of ℓ-highest weight nor of ℓ-lowest weight, and not in the category $\mathcal{O}$, is given in [Her09]. The goal of our work is to construct a new family of extremal loop weight modules.

1.7.2 Case of the simple ℓ-highest weight representations

Proposition 1.7.2. *[Man13a] Let m be a dominant monomial. The simple ℓ-highest weight module $V(m)$ of ℓ-highest weight m is an extremal loop weight module of $\mathcal{U}_q(sl_{n+1}^{tor})$.*

Proof. Let m be a dominant monomial. We have viewed that $V(m)$ is an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module which satisfies $V(m) = \mathcal{U}_q(sl_{n+1}^{tor}) \cdot v$ (with v the ℓ-highest weight vector of $V(m)$). Let us show that v is an extremal vector of weight $\omega(m)$. As v is of ℓ-highest weight, it is in particular a highest weight vector of weight $\omega(m)$. So it satisfies for all $i \in I$

$$(x_i^-)^{1+\omega(m)(h_i)} \cdot v = 0.$$

These relations define the simple highest weight $\mathcal{U}_q(\hat{sl}_{n+1})$-module $V(\omega(m))$. Hence we have a surjective $\mathcal{U}_q(\hat{sl}_{n+1})$-morphism $V(\omega(m)) \twoheadrightarrow \mathcal{U}_q^h(sl_{n+1}^{tor}) \cdot v$. As $V(\omega(m))$ is simple, this is an isomorphism of $\mathcal{U}_q(\hat{sl}_{n+1})$-modules, and v is extremal for the horizontal quantum affine subalgebra of weight $\omega(m)$.

It remains to show that for all $w \in V(m)$ and $j \in I$, $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot w$ is finite-dimensional. Actually, one can assume that $j = 0$ in our computation. In fact if we prove the property for the vertical quantum affine subalgebra $\mathcal{U}_q^v(sl_{n+1}^{tor})$, the third point of Definition 1.7.1 follows by application of $\theta^{(j)}$.

So fix $w \in V(m)$ an ℓ-weight vector and consider the sub-$\mathcal{U}_q^v(sl_{n+1}^{tor})$-module $W = \mathcal{U}_q^v(sl_{n+1}^{tor}) \cdot w$ of $V(m)$. Denote by λ the weight of w and $k = \lambda(d)$. As the weight lattice of $\mathcal{U}_q^v(sl_{n+1}^{tor})$ is $P_0 = \mathbb{Z}\Lambda_1 + \cdots + \mathbb{Z}\Lambda_n$, we have

$$\mathrm{wt}(W) \subseteq \{\mu \in \mathrm{wt}(V(m)) | \mu(d) = k\}.$$

Further the weight spaces of $V(m)$ and W are finite-dimensional. So it suffices to show that the set of weights $\{\mu \in \mathrm{wt}(V(m)) | \mu(d) = k\}$ is finite. For that we use the theory of q-characters.

Let us recall some facts about the q–character $\chi_q(V(m))$ of the simple ℓ-highest weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(m)$. By [Her07, Proposition 6.1] it is a sub-factor of a product of q–characters of fundamental representations. So we are reduced to prove that

$$\{\mu \in \mathrm{wt}(V(Y_{\ell,a})) | \mu(d) = k\}$$

is finite for all $k \in \mathbb{Z}$ and all $\ell \in I, a \in \mathbb{C}^*$. But the q–characters $\chi_q(V(Y_{\ell,a}))$ (and in particular there characters) are known, given in terms of tableaux sum expressions (see [Her11]). And we check directly from these formulas the finiteness of $\{\mu \in \mathrm{wt}(V(Y_{\ell,a})) | \mu(d) = k\}$. Hence $\mathcal{U}_q^v(sl_{n+1}^{tor}) \cdot w$ is finite-dimensional, and the simple ℓ-highest weight module $V(m)$ is an extremal loop weight module. □

1.8 Drinfeld coproduct

Let Δ_D be the coproduct defined by $(i \in I)$

$$\Delta_D(x_i^+(z)) = x_i^+(z) \otimes 1 + \phi_i^-(z) \otimes x_i^+(z), \tag{1.6}$$

$$\Delta_D(x_i^-(z)) = x_i^-(z) \otimes \phi_i^+(z) + 1 \otimes x_i^-(z), \tag{1.7}$$

$$\Delta_D(\phi_i^\pm(z)) = \phi_i^\pm(z) \otimes \phi_i^\pm(z). \tag{1.8}$$

It takes part in the whole Chapter 3 of the thesis. Note that this map does not define a coproduct in the usual sense because (4.5) and (4.6) involve infinite sums and are not elements of $\mathcal{U}_q(sl_{n+1}^{tor}) \otimes \mathcal{U}_q(sl_{n+1}^{tor})$. However it can be used to define a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module on (a subspace of) the tensor product of $\mathcal{U}_q(sl_{n+1}^{tor})$-modules when the summations are well-defined. In fact, we have

Lemma 1.8.1. *[FJMM13] Let V, W be $\mathcal{U}_q(sl_{n+1}^{tor})$-modules. Let $U \subseteq V \otimes W$ be a linear subspace such that for any $u \in U$ and $i \in I$, $\Delta_D(x_i^+(z)) \cdot u$ is well-defined in U. Then the coproduct Δ_D endows U with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module.*

We will have to consider another situation: assume that we have a decomposition of the vector space $V \otimes W = U \oplus U'$ such that $\phi_i^\pm(z) \cdot U \subset U$ and $\phi_i^\pm(z) \cdot U' \subset U'$ for all $i \in I$. Let us denote by $\mathrm{pr}_U : U \oplus U' \to U$ and $\mathrm{pr}_{U'} : U \oplus U' \to U'$ the projections along U and U' respectively. Assume that the action of $x_i^\pm(z)$ is well-defined on the vectors in U. Assume further that the vector spaces U and U' have bases $\{e_r\}_{r \in R}$ and $\{f_s\}_{s \in S}$ respectively such that the matrix coefficients of operators $x_i^\pm(z)$ applied to f_s and computed on e_r are zero for all $s \in S$ and $r \in R$. Then we have

Lemma 1.8.2. *[FJMM13] Under the assumption above, the space U has a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module such that the series $x_i^\pm(z)$ and $\phi_i^\pm(z)$ act by $(\mathrm{pr}_U \otimes \mathrm{pr}_U) \circ \Delta_D(x_i^\pm(z))$ and $(\mathrm{pr}_U \otimes \mathrm{pr}_U) \circ \Delta_D(\phi_i^\pm(z))$ respectively.*

One can improve these results as follows.

Lemma 1.8.3. *[Man13a] Assume further that V and W are ℓ-integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-modules in the last lemmas. Then U is an ℓ-integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module.*

Proof. As V and W have an ℓ-weight space decomposition and by (4.7), we have $U = \oplus_{m \in A} U_m$ with for all $m \in A$,

$$U_m = \bigoplus_{m = m' \times m''} V_{m'} \otimes W_{m''}.$$

Let $u = v \otimes w \in U$ and fix $i \in I$. As V and W are ℓ-integrable, the subspaces $\hat{\mathcal{U}}_i \cdot v$, $\hat{\mathcal{U}}_i \cdot w$ are finite-dimensional. By (4.5), (4.6) and (4.7) we have

$$\hat{\mathcal{U}}_i \cdot u \subseteq (\hat{\mathcal{U}}_i \cdot v) \otimes (\hat{\mathcal{U}}_i \cdot w).$$

Hence $\hat{\mathcal{U}}_i \cdot u$ is finite-dimensional, and U is an ℓ-integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module. $\square$

Remark 1.8.4. *Assume that for all $v \in V, w \in W$ and all $j \in I$, $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot v$ and $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot w$ are finite-dimensional. We show as in the last lemma that the $\mathcal{U}_q(sl_{n+1}^{tor})$-module U satisfies also*

$$\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot u \text{ is finite-dimensional for all } u \in U \text{ and } j \in I.$$

Chapter 2

Extremal loop weight modules and monomial crystals

This chapter corresponds to the paper [Man12c], to appear in *Pacific Journal of Mathematics.*

2.1 Motivations

2.1.1 Abstract

In this chapter we construct a new family of representations for the quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$, which are ℓ-extremal in the sense of Hernandez [Her09]. We construct extremal loop weight modules associated to level 0 fundamental weights ϖ_ℓ when $n = 2r+1$ is odd and $\ell = 1, r+1$ or n. To do it, we relate monomial realizations of level 0 extremal fundamental weight crystals to integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$, and we introduce promotion operators for the level 0 extremal fundamental weight crystals. By specializing the quantum parameter, we get finite-dimensional modules of quantum toroidal algebras at roots of unity. In general, we give a conjectural process to construct extremal loop weight modules from monomial realizations of crystals.

2.1.2 Motivations

In [Kas03, Nak03], Kashiwara and Nakajima have defined a crystal $\mathcal{M}$, called the monomial crystal, whose vertices are Laurent monomials. They determined monomial realizations of crystals of finite type. These results have been extended in [HN06] to the level 0 extremal weight $\mathcal{U}_q(\hat{sl}_{n+1})$-crystals $\mathcal{B}(\varpi_\ell)$ $(1 \leq \ell \leq n)$: if $n = 2r + 1$ is odd, it is isomorphic to the sub-$\mathcal{U}_q(\hat{sl}_{n+1})$-crystal $\mathcal{M}_\ell$ of $\mathcal{M}$.

The monomials occurring in these realizations of crystals can be interpreted at the level of representation theory. In fact Frenkel and Reshetikhin [FR99] defined a correspondence between ℓ-weights (eigenvalues of the Cartan subalgebra for the Drinfeld realization) and these monomials. Motivated by these facts, Hernandez [Her09] used the monomial $\mathcal{U}_q(\hat{sl}_4)$-crystal $\mathcal{M}_1$ to construct an integrable representation of $\mathcal{U}_q(sl_4^{tor})$ whose ℓ-weights are the monomials occurring in this crystal. He defined in this way the first example of extremal loop weight modules for $\mathcal{U}_q(sl_4^{tor})$.

We use the same technical feature in this chapter. We propose to relate the monomial $\mathcal{U}_q(\hat{sl}_{n+1})$-crystals $\mathcal{M}_\ell$ (where $n = 2r + 1$ is supposed to be odd and $\ell \in I_0$) with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$. The aim is the construction of extremal loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-modules and, by specialization of the quantum parameter, of finite-dimensional representations of the quantum toroidal algebras at roots of unity.

2.1.3 Summary of results

In this chapter, we relate the monomial realizations $\mathcal{M}_\ell$ of the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ (where $n = 2r + 1$ is supposed to be odd and $\ell \in I_0$) with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$. More precisely, the aim is to construct a $\mathcal{U}_q(sl_{n+1}^{tor})$-module whose the q–character is the sum of monomials occurring in $\mathcal{M}_\ell$.

A monomial set is not in general the set of ℓ-weights of an integrable representation: it must satisfy combinatorial properties related to the theory of q-characters (see [FM01, FR99]). This leads us to introduce the notion of closed monomial set.

Definition. *(Definition 2.2.6)*

(i) A set of monomials $\mathcal{S} \subset \mathcal{M}$ is said to be q-closed in the direction i ($i \in I$) if for all $m \in \mathcal{S}$ there exists a finite subset

$$\mathcal{S}_m \subset \mathcal{S} \cap \left(m \cdot \prod_{l \in \mathbb{Z}} A_{i,l}^{\mathbb{Z}} \right)$$

containing m and a sequence $(n_s)_{s \in \mathcal{S}_m}$ of positive integers such that $\Xi_i \left(\sum_{s \in \mathcal{S}_m} n_s \cdot s \right)$ is the q–character of a representation of $\hat{\mathcal{U}}_i$.

(ii) A set of monomials $\mathcal{S}$ is said to be J-q-closed ($J \subset I$), or simply q-closed if $J = I$, if $\mathcal{S}$ is q-closed in the direction i for all $i \in J$.

(iii) A set of monomials $\mathcal{S} \subset \mathcal{M}$ is said to be J-closed by the Kashiwara operators ($J \subset I$), or simply closed by the Kashiwara operators if $J = I$, if the operators $\tilde{e}_i$, $\tilde{f}_i$ preserve $\mathcal{S}$ for all $i \in J$.

(iv) A set of monomials $\mathcal{S} \subset \mathcal{M}$ which is J-q-closed and J-closed by the Kashiwara operators ($J \subset I$), is called a J-closed set. If $J = I$, it is simply called a closed monomial set.

It gives a necessary condition for a set to be the set of ℓ-weights of an integrable representation.

Let us consider the monomial realizations $\mathcal{M}_\ell$ of the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ ($\ell \in I_0$): this is the sub-$\mathcal{U}_q(\hat{sl}_{n+1})$-crystal of $\mathcal{M}$ generated by the monomial $e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1}$. We determine when the monomial crystal $\mathcal{M}_\ell$ is closed.

Proposition. *(Theorem 2.2.22) The monomial crystal $\mathcal{M}_\ell$ is closed if and only if*

$$\ell = 1, r+1 \text{ or } n.$$

The proof is based on the combinatorial study of these crystals: the cyclic symmetry of the Dynkin diagram of type $A_n^{(1)}$ has a counterpart at the level of crystals. Actually, these symmetry properties are already known for the $\mathcal{U}_q(sl_{n+1})$-crystals of finite type, and translated into the existence of promotion operators pr (see [BST10, FOS09, OS08, Sch08, Shi02] and references therein). Here we introduce promotion operators for the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ $(1 \leq \ell \leq n)$.

When $\mathcal{M}_\ell$ is closed, we construct an associated integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module whose the set of ℓ-weights consists of monomials occurring in $\mathcal{M}_\ell$.

Theorem. *(Theorem 2.3.1) Assume that $n = 2r+1$ is odd and $\ell = 1, r+1$ or n. There exists a thin representation V_ℓ of $\mathcal{U}_q(sl_{n+1}^{tor})$ whose q–character is the sum of all monomials occurring in $\mathcal{M}_\ell$ with multiplicity one. It is called the extremal fundamental loop weight module.*

To construct the representation V_ℓ, we paste together some finite-dimensional representations of the various vertical quantum affine subalgebas of $\mathcal{U}_q(sl_{n+1}^{tor})$. The existence of promotion operators for $\mathcal{M}_\ell$ involves that it defines a $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure.

One of the main result of this chapter is the following.

Theorem. *(Theorem 2.3.7) The $\mathcal{U}_q(sl_{n+1}^{tor})$-module V_ℓ is an extremal loop weight module of ℓ-weight $e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1}$.*

Furthermore we show that V_ℓ is irreducible, and is isomorphic to the level 0 fundamental extremal representations $V(\varpi_\ell)$ as modules over the horizontal quantum affine subalgebra.

When the monomial crystal $\mathcal{M}_\ell$ is not closed, there is no integrable representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ whose set of ℓ-weights consists of monomials occurring in it. The idea is to consider instead of $\mathcal{M}_\ell$ a closed crystal $\overline{\mathcal{M}_\ell}$ containing it and to apply the preceding methods to this crystal.

We treat an example of such a construction: set $M = e^{2\varpi_1} Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1}$. We consider the sub-$\mathcal{U}_q(\hat{sl}_{n+1})$-crystal $\mathcal{M}(M)$ of $\mathcal{M}$ generated by the monomial M. It does not satisfy the definition of closed monomial set. We determine a closed monomial set $\overline{\mathcal{M}}(M)$ containing it on which we apply the preceding methods.

Theorem. *(Theorem 2.4.6) There exists a thin representation of $\mathcal{U}_q(sl_4^{tor})$ whose q–character is the sum of monomials occurring in $\overline{\mathcal{M}}(M)$ with multiplicity one. It is denoted by $V(M)$.*

The construction is analogous to the one of the extremal fundamental loop weight modules: we paste together finite-dimensional representations of vertical quantum affine subalgebras of $\mathcal{U}_q(sl_4^{tor})$. Furthermore we have

Theorem. *(Theorem 2.4.11) The representation $V(M)$ is an extremal loop weight module of ℓ-weight M.*

In addition to the promotion operators, there exists shift automorphisms of the spectral parameter $(a \in \mathbb{C}^*)$

$$\tau_{p_\ell} : Y_{i,a}^{\pm} \mapsto Y_{i,aq^{p_\ell}}^{\pm}$$

on the monomial crystals $\mathcal{M}_\ell$. These shift automorphisms are related to finite-dimensional representations of the quantum toroidal algebra at roots of unity. This is the main motivation of our study of extremal loop weight modules. We get an important result in this way.

Theorem. *(Theorem 2.3.18) Set $\ell = 1, n$ and $p_\ell = n+1$ (resp. $\ell = r+1$ and $p_\ell = 2$). Denote by ϵ a primitive $[p_\ell L]$-root of unity with $L \geq 1$ (resp. $L > 1$). There is an irreducible $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$-module $[V_{\ell,a}]_\epsilon$ of dimension $L \times \begin{pmatrix} n+1 \\ \ell \end{pmatrix}$.*

2.1.4 Organization

Let us now describe briefly the organization of this chapter.

Section 2.2 is devoted to the study of monomial crystals. We recall its definition and we introduce the notion of closed monomial set (Definition 2.2.6). We introduce promotion operators for the level 0 fundamental extremal weight crystals. As a consequence, we determine when $\mathcal{M}_\ell$ is closed (Theorem 2.2.22).

In Section 2.3 we construct a new family of representations of $\mathcal{U}_q(sl_{n+1}^{tor})$ (the extremal fundamental loop weight modules) when n is odd and $\mathcal{M}_\ell$ is closed (Theorem 2.3.1). We check that these representations satisfy the definition of extremal loop weight modules (Theorem 2.3.7) and we give formulas for the action (Theorem 2.3.12). We get finite-dimensional representations of the quantum toroidal algebra at roots of unity by specializing the quantum parameter q at roots of unity (Theorem 2.3.18).

In Section 2.4 we treat an example where the considered monomial crystal is not closed. We construct a representation of $\mathcal{U}_q(sl_4^{tor})$ associated to the level 0 weight $2\varpi_1$.

2.2 Study of the monomial crystals $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$

We will relate in this chapter the monomial realizations $\mathcal{M}_\ell$ of level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ $(1 \leq \ell \leq n)$ of $\mathcal{U}_q(\hat{sl}_{n+1})$ with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$. In this section, we study the combinatorics of these monomial realizations, the main point being the use of promotion operators for level 0 extremal fundamental weight crystals introduced below. This is the first step of the construction of integrable modules associated to $\mathcal{M}_{\ell,a}$.

In the first subsection, one gives the definition of the monomial $\mathcal{U}_q(\hat{sl}_{n+1})$-crystal $\mathcal{M}$ [Kas03, Nak03]. This definition holds when the considered Cartan matrix has no odd cycle. So it does not work for $\mathcal{U}_q(\hat{sl}_{n+1})$ when n is even, and we have to assume that $n = 2r + 1$ $(r \geq 1)$ is odd until the end of Chapter 2. Following [HN06], we recall the monomial realization of $\mathcal{B}(\varpi_\ell)$ $(1 \leq \ell \leq n)$: it is isomorphic to the sub-$\mathcal{U}_q(\hat{sl}_{n+1})$-crystal $\mathcal{M}_\ell = \mathcal{M}(e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1})$ of $\mathcal{M}$ generated by the monomial $e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1}$ (with d_ℓ equal to $\min(\ell, n+1-\ell)$). Furthermore we define the notions of q-closed monomial set and of monomial set closed by the Kashiwara operators, respectively related to the theory of q-characters and to the combinatoric of crystals.

The monomial crystals $\mathcal{M}_\ell$ are already studied in [HN06]: the monomials occurring in these crystals are explicitly given for $1 \leq \ell \leq n$ and their automorphisms z_ℓ are described in terms of monomials. We recall these results in the second subsection.

In the third subsection, we introduce promotion operators for level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ $(1 \leq \ell \leq n)$. The promotion operators were introduced in [Shi02] for the Young tableaux realization of the finite $\mathcal{U}_q(sl_{n+1})$-crystals $\mathcal{B}_0(k\Lambda_\ell)$ $(k \in \mathbb{N}^*$, $1 \leq \ell \leq n)$ and studied in numerous papers (see [BST10, FOS09, OS08, Sch08, Shi02] and references therein). It is the counterpart at the level of crystals of the cyclic symmetry of the Dynkin diagram of type $A_n^{(1)}$. After recalling these definitions, we extend the promotion operators for the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$. Finally we specify the promotion operator of $\mathcal{B}(\varpi_\ell)$ in its monomial realization $\mathcal{M}_\ell$.

In the last subsection, we use promotion operators to obtain a new description of $\mathcal{M}_\ell$. In particular, we improve results given in [HN06] for these crystals. Furthermore we determine the $\ell \in I_0$ for which the monomial crystals $\mathcal{M}_\ell$ are closed (Theorem 2.2.22): this is the case if and only if $\ell = 1, r+1$ or n.

2.2.1 Monomial crystals

In this section we define the monomial crystal $\mathcal{M}$ of $\mathcal{U}_q(\hat{sl}_{n+1})$ when $n = 2r+1$ is supposed to be odd, following [Kas03, Nak03]. Monomial realizations of the crystals $\mathcal{B}(\lambda)$ with $\lambda \in P$, in particular of $\mathcal{B}(\varpi_\ell)$ $(1 \leq \ell \leq n)$, are studied in [HN06, Kas03, Nak03]. They are obtained as subcrystals of $\mathcal{M}$ generated by a monomial. We recall these results here. Finally we introduce new notions of q-closed monomial set and of monomial set closed by the Kashiwara operators.

As we have said above, the definition of the monomial crystal $\mathcal{M}$ requires that the considered Cartan matrix C is without odd cycle. So we assume that $n = 2r+1$ $(r \geq 1)$ is odd until the end of this chapter. In particular there is a function $s : I \to \{0,1\}, i \mapsto s_i$ such that $C_{i,j} = -1$ implies $s_i + s_j = 1$.

Let us recall that $A_{\mathbb{Z}}$ is the group of monomials of the form $m = e^{\omega(m)} \prod_{i\in I, l\in\mathbb{Z}} Y_{i,q^l}^{u_{i,q^l}(m)}$ where $u_{i,q^l}(m) \in \mathbb{Z}$, $\omega(m) \in P$ are such that

$$\sum_{l\in\mathbb{Z}} u_{i,q^l}(m) = \omega(m)(h_i).$$

To simplify the notations, we will denote $Y_{i,q^l}^{\pm 1}$ and $u_{i,q^l}(m)$ only by $Y_{i,l}^{\pm 1}$ and $u_{i,l}$ in the whole Chapter 2 $(i \in I, l \in \mathbb{Z})$.

Consider the subgroup $\mathcal{M} \subset A_{\mathbb{Z}}$ defined by

$$\mathcal{M} = \{m \in A_{\mathbb{Z}} \mid u_{i,l}(m) = 0 \text{ if } l \equiv s_i + 1 \bmod 2\}.$$

Following [Kas03, Nak03], let us define $\mathrm{wt} : \mathcal{M} \to P$ and

$$\varepsilon_i, \varphi_i, p_i, q_i : \mathcal{M} \to \mathbb{Z} \cup \{\infty\} \cup \{-\infty\}$$

for $i \in I$ by $(m \in \mathcal{M})$

$$\begin{array}{c}
\mathrm{wt}(m) = \omega(m), \\
\varphi_{i,L}(m) = \sum_{l\leq L} u_{i,l}(m),\ \varphi_i(m) = \max\{\varphi_{i,L}(m) | L \in \mathbb{Z}\} \geq 0, \\
\varepsilon_{i,L}(m) = -\sum_{l\geq L} u_{i,l}(m),\ \varepsilon_i(m) = \max\{\varepsilon_{i,L}(m) | L \in \mathbb{Z}\} \geq 0,
\end{array}$$

$$\begin{aligned}
p_i(m) &= \max\{L \in \mathbb{Z} | \varepsilon_{i,L}(m) = \varepsilon_i(m)\} \\
&= \max\left\{L \in \mathbb{Z} \,\middle|\, \sum_{l<L} u_{i,l}(m) = \varphi_i(m)\right\}, \\
q_i(m) &= \min\{L \in \mathbb{Z} | \varphi_{i,L}(m) = \varphi_i(m)\} \\
&= \min\left\{L \in \mathbb{Z} \,\middle|\, -\sum_{l>L} u_{i,l}(m) = \varepsilon_i(m)\right\}.
\end{aligned}$$

Then we define $\tilde{e}_i, \tilde{f}_i : \mathcal{M} \to \mathcal{M} \cup \{0\}$ for $i \in I$ by

$$\tilde{e}_i \cdot m = \begin{cases} 0 & \text{if } \varepsilon_i(m) = 0, \\ mA_{i,p_i(m)-1} & \text{if } \varepsilon_i(m) > 0, \end{cases}$$

$$\tilde{f}_i \cdot m = \begin{cases} 0 & \text{if } \varphi_i(m) = 0, \\ mA_{i,q_i(m)+1}^{-1} & \text{if } \varphi_i(m) > 0, \end{cases}$$

where $A_{i,l} = e^{\alpha_i} Y_{i,l-1} Y_{i,l+1} Y_{i-1,l}^{-1} Y_{i+1,l}^{-1} \in A_{\mathbb{Z}}$.

Theorem 2.2.1. *[Kas03, Nak03] $(\mathcal{M}, \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ is a $\mathcal{U}_q(\hat{sl}_{n+1})$–crystal, called the monomial crystal.*

Remark 2.2.2. *When n is even, the Dynkin diagram of type $A_n^{(1)}$ is not bipartite. In this case, $(\mathcal{M}, \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ does not satisfy the axioms of crystal (see [Kas03]). Another crystal structures are defined on (a subset of) $A_{\mathbb{Z}}$ in [Kas03]. But the monomials used are different with those occurring in the theory of q–characters of $\mathcal{U}_q(sl_{n+1}^{tor})$-modules and it is not useful for what we will do in this chapter.*

For $m \in \mathcal{M}$ denote by $\mathcal{M}(m)$ the connected subcrystal of $\mathcal{M}$ generated by m. As it is explained above, the weight of a monomial $m' \in \mathcal{M}(m)$ is determined by $\omega(m)$ and $u_{i,l}(m')$ (Remark 1.6.8). So we will omit the term $e^{\omega(m')}$ and we just specify the weight of the monomial m. For $J \subset I$ and $m \in \mathcal{M}$, denote by $\mathcal{M}_J(m)$ the set of monomials obtained from m by applying the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ for $i \in J$. It is a connected sub-J-crystal of $\mathcal{M}(m)$ generated by m.

For $p \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}\delta$, let $\tau_{2p,\alpha}$ be the map $\tau_{2p,\alpha} \colon \mathcal{M} \to \mathcal{M}$ defined by

$$\tau_{2p,\alpha}(e^{\lambda} \prod Y_{i,n}^{u_{i,n}}) = e^{\lambda+\alpha} \prod Y_{i,n+2p}^{u_{i,n}}.$$

This is a P_{cl}-crystal automorphism of the crystal $\mathcal{M}$, also called shift automorphism in the following.

The following result was proved in [Kas03, Nak03] when m is a dominant monomial and is generalized in [HN06] for all $m \in \mathcal{M}$.

Theorem 2.2.3. *For $m \in \mathcal{M}$, the crystal $\mathcal{M}(m)$ is isomorphic to a connected component of the crystal $\mathcal{B}(\lambda)$ of an extremal weight module for some $\lambda \in P$.*

It was shown in [Kas02b] that the fundamental extremal crystals $\mathcal{B}(\varpi_\ell)$ are connected for all $\ell \in I_0$. Let

$$d_\ell = \min(\ell, n+1-\ell)$$

be the distance between the nodes 0 and ℓ in the Dynkin diagram of type $A_n^{(1)}$. We have the following monomial realization of $\mathcal{B}(\varpi_\ell)$.

Theorem 2.2.4. *[HN06] Set $M = e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1}$ for $\ell \in I_0$. Then M is extremal in $\mathcal{M}$ and $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$ as P-crystals.*

One can define in the same way the monomial crystal $\mathcal{M}_0$ associated to $\mathcal{U}_q(sl_{n+1})$. It can be done for all $n \geq 2$, the Cartan matrix of type A_n being without cycle. As it is said above, the weights of monomials are completely determined by the powers of variables $Y_{i,l}^{\pm 1}$ in this case. So they can be safely omitted. For $m \in \mathcal{M}_0$, we denote by $\mathcal{M}_0(m)$ the subcrystal of $\mathcal{M}_0$ generated by m. We have

Proposition 2.2.5. *[Kas03, Nak03] The $\mathcal{U}_q(sl_{n+1})$-crystals $\mathcal{M}_0(Y_{i,k})$ and $\mathcal{B}_0(\Lambda_i)$ are isomorphic for all $i \in I_0$ and $k \in \mathbb{Z}$.*

For $i \in I$, set $\Xi_i : \mathcal{M} \to \mathcal{M}$ the map sending the variables $Y_{j,*}^{\pm 1}, e^{\nu}$ to 1 for all $j \neq i$ and $\nu \in P$, and $Y_{i,*}^{\pm 1}$ to themselves. Another map will be used below: $\Xi^i : \mathcal{M} \to \mathcal{M}$ which sends the variables $Y_{j,*}^{\pm 1}$ to themselves if $j \neq i$ and $Y_{i,*}^{\pm 1}, e^{\nu}$ to 1 for all $\nu \in P$. These two maps are also defined in [FM01] and denoted by $\beta_{\{i\}}$ and β_{I_i} respectively.

Definition 2.2.6. *(i) A set of monomials $\mathcal{S} \subset \mathcal{M}$ is said to be q-closed in the direction i ($i \in I$) if for all $m \in \mathcal{S}$ there exists a finite subset*

$$\mathcal{S}_m \subset \mathcal{S} \cap \left(m \cdot \prod_{l \in \mathbb{Z}} A_{i,l}^{\mathbb{Z}} \right)$$

containing m and a sequence $(n_s)_{s \in \mathcal{S}_m}$ of positive integers such that $\Xi_i \left(\sum_{s \in \mathcal{S}_m} n_s \cdot s \right)$ is the q–character of a representation of $\hat{\mathcal{U}}_i$.

(ii) A set of monomials $\mathcal{S}$ is said to be J-q-closed ($J \subset I$), or simply q-closed if $J = I$, if $\mathcal{S}$ is q-closed in the direction i for all $i \in J$.

(iii) A set of monomials $\mathcal{S} \subset \mathcal{M}$ is said to be J-closed by the Kashiwara operators ($J \subset I$), or simply closed by the Kashiwara operators if $J = I$, if the operators $\tilde{e}_i$, $\tilde{f}_i$ preserve $\mathcal{S}$ for all $i \in J$.

(iv) A set of monomials $\mathcal{S} \subset \mathcal{M}$ which is J-q-closed and J-closed by the Kashiwara operators ($J \subset I$), is called a J-closed set. If $J = I$, it is simply called a closed monomial set.

Remark 2.2.7. *(i) The definition of q-closed set is inspired by the theory of q–characters and the Frenkel-Mukhin algorithm [FM01]. In particular, it involves q-characters of $\mathcal{U}_q(\hat{sl}_2)'$-modules. Let us recall that in this case, the image of $\chi_q : \mathcal{R}_{l,\mathbb{Z}} \to \mathbb{Z}[Y_{1,l}^{\pm 1}]_{l \in \mathbb{Z}}$ is known (see [FR99]): it is equal to the subring $\mathbb{Z}[(Y_{1,l} + Y_{1,l+2}^{-1})]_{l \in \mathbb{Z}}$ of $\mathbb{Z}[Y_{1,l}^{\pm 1}]_{l \in \mathbb{Z}}$ generated by the $Y_{1,l} + Y_{1,l+2}^{-1}$ ($l \in \mathbb{Z}$).*

(ii) The notion of q-closed set holds also for the monomial $\mathcal{U}_q(sl_{n+1})$-crystal $\mathcal{M}_0$. Further it extends naturally when q is specialized at roots of unity, by using the theory of q–characters at roots of unity [FM02b].

Let V be an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module such that for all ℓ-weight (ν, γ) of V, $V_{(\nu,\gamma)}$ is finite-dimensional and the roots of the associated polynomials $Q_i(z)$ and $R_i(z)$ are in $q^{\mathbb{Z}}$ for all $i \in I$. Then the monomial set $\mathcal{M}(V)$ is q-closed. Note that it is not necessary that the Frenkel-Mukhin algorithm holds for V: for example it does not work for the simple finite-dimensional $\mathcal{U}_q(\hat{sl}_3)'$-module $V_0(Y_{1,0}^2 Y_{2,3}) \simeq V_0(Y_{1,0} Y_{2,3}) \otimes V_0(Y_{1,0})$ considered in [HL10], but $\mathcal{M}(V_0(Y_{1,0}^2 Y_{2,3}))$ is q-closed.

In general, $\mathcal{M}(V)$ is not closed by the Kashiwara operators: for example the q–character of the $\mathcal{U}_q(\hat{sl}_2)'$-module $V_0(Y_{1,2} Y_{1,0}^2)$ contains the monomial $Y_{1,0}$ but does not contain $Y_{1,2}^{-1}$. However, it holds for the fundamental $\mathcal{U}_q(\hat{sl}_{n+1})'$-modules. In fact by using the tableaux sum expressions of their q–characters given in [Nak03], we have

Proposition 2.2.8. *[Nak03] Let $V_0(Y_{i,k})$ be a fundamental representation of $\mathcal{U}_q(\hat{sl}_{n+1})'$ ($i \in I_0$, $k \in \mathbb{Z}$). Then the monomial sets $\mathcal{M}_0(Y_{i,k})$ and $\mathcal{M}(V_0(Y_{i,k}))$ are equal.*

In particular by Proposition 2.2.5, $\mathcal{M}(V_0(Y_{i,k}))$ has a $\mathcal{U}_q(sl_{n+1})$-crystal structure isomorphic to $\mathcal{B}_0(\Lambda_i)$. As a consequence we have

Corollary 2.2.9. *For all $1 \leq i \leq n$ and $k \in \mathbb{Z}$, the $\mathcal{U}_q(sl_{n+1})$-crystal $\mathcal{M}_0(Y_{i,k})$ is closed.*

Finally, let us give an example of monomial crystal which is not q-closed. Consider the $\mathcal{U}_q(sl_2)$-crystal $\mathcal{M}_0(Y_{1,4} Y_{1,0})$

$$Y_{1,4} Y_{1,0} \to Y_{1,6}^{-1} Y_{1,0} \to Y_{1,6}^{-1} Y_{1,2}^{-1}.$$

If $\mathcal{M}_0(Y_{1,4} Y_{1,0})$ is q-closed, it should contain $\mathcal{M}(V_0(Y_{1,4} Y_{1,0}))$. This is not the case, the q-character of $V_0(Y_{1,4} Y_{1,0})$ being

$$\chi_q(V_0(Y_{1,4} Y_{1,0})) = Y_{1,4} Y_{1,0} + Y_{1,6}^{-1} Y_{1,0} + Y_{1,4} Y_{1,2}^{-1} + Y_{1,6}^{-1} Y_{1,2}^{-1}.$$

2.2.2 Description of the monomial crystal $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$

Assume that $n = 2r+1$ is odd with $r \geq 1$. The monomial crystals $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ are studied in [HN06, Section 4]: the monomials occurring in these crystals are explicitly described and the automorphisms z_ℓ are given in terms of monomials. We recall these results here.

To describe the monomial crystals $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$, we assume in this section that $\ell \leq r+1$ (as in [HN06]). Let us begin by explaining why we can do that. We need the notion of twisted isomorphism of crystals (this definition appears in [BST10]).

Definition 2.2.10. *Let $\mathcal{B}$ and $\mathcal{B}'$ be crystals over two isomorphic Dynkin diagrams D and D' with vertices respectively indexed by I and I' and let $\theta : I \to I'$ be an isomorphism from D to D'. Then ϕ is a θ-twisted isomorphism if $\phi : \mathcal{B} \to \mathcal{B}'$ is a bijection map and for all $b \in \mathcal{B}$ and $i \in I$,*

$$\tilde{f}_{\theta(i)} \cdot \phi(b) = \phi(\tilde{f}_i \cdot b) \text{ and } \tilde{e}_{\theta(i)} \cdot \phi(b) = \phi(\tilde{e}_i \cdot b).$$

Let ι be the automorphism of the Dynkin diagram of type $A_n^{(1)}$ such that $\iota(k) = -k$ ($k \in I$) where I is identified to the set $\mathbb{Z}/(n+1)\mathbb{Z}$. It defines an automorphism $\iota_{\mathfrak{h}}$ of $\mathfrak{h}$ by sending h_i, d to $h_{\iota(i)}, d$ for all $i \in I$. Let $\psi : \mathcal{M} \to \mathcal{M}$ be the map defined by ($r \in \mathbb{Q}$)

$$\psi\left(e^{r\delta} \prod (e^{\Lambda_i} Y_{i,n})^{u_{i,n}}\right) = e^{r\delta} \prod (e^{\Lambda_{-i}} Y_{-i,n})^{u_{i,n}}.$$

Then we show easily that ψ is a ι-twisted automorphism of the P-crystal $\mathcal{M}$. Furthermore it induces a ι-twisted isomorphism

$$\psi : \mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1}) \longrightarrow \mathcal{M}(e^{\varpi_{n+1-\ell}} Y_{n+1-\ell,0} Y_{0,\ell}^{-1})$$

between the monomial crystals $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$ and $\mathcal{M}(e^{\varpi_{n+1-\ell}} Y_{n+1-\ell,0} Y_{0,\ell}^{-1})$ for all $1 \leq \ell \leq r+1$.

So one can assume that $1 \leq \ell \leq r+1$. In this case, $d_\ell = \ell$ and we study the crystal $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$ (see [HN06]). One defines the monomials

$$\boxed{k}_p = Y_{k-1,p+k}^{-1} Y_{k,p+k-1} \qquad \text{for } 1 \leq k \leq n+1,\ p \in \mathbb{Z}.$$

with $Y_{n+1,p} = Y_{0,p}$ by convention. By Remark 1.6.8, the terms $e^{\omega(m')}$ can be safely omitted for all $m' \in \mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$. Set $M_0 = e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1}$ and

$$\begin{aligned} M_j &= Y_{\ell,2j} Y_{0,n-\ell+1+2j}^{-1} Y_{j,\ell+j}^{-1} Y_{j,n-\ell+1+j} \\ &= \left(\boxed{1}_{n-\ell+2j} \boxed{2}_{n-\ell+2j-2} \cdots \boxed{j}_{n-\ell+2}\right) \times \left(\boxed{j{+}1}_{\ell-1} \boxed{j{+}2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell+2j}\right) \\ &= \prod_{p=1}^{j} \boxed{p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{p}_{\ell+1-2p+2j} \end{aligned}$$

with $0 \leq j \leq \ell$. In particular, $M_\ell = Y_{\ell,n+1} Y_{0,n+1+\ell}^{-1} = \tau_{n+1,-\ell\delta}(M_0)$ and $M_1 = \tau_{2,-\delta}(M_0)$ for $\ell = r+1$. One defines other monomials in the following way: for $j \in \mathbb{Z}$ and a Young tableau of shape (ℓ) $T = (1 \leq i_1 < i_2 < \cdots < i_\ell \leq n+1)$ we set

$$m_{T;j} = \prod_{p=1}^{j} \boxed{i_p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{i_p}_{\ell+1-2p+2j} \qquad \text{for } 0 \leq j \leq \ell-1, \tag{2.1}$$

and $m_{T;j+\ell} = \tau_{n+1,-\ell\delta}(m_{T;j})$. Note that $M_j = m_{T;j}$ with $T = (1, 2, \ldots, \ell)$.

By Theorem 2.2.4, $\mathcal{M}(M_0)$ and $\mathcal{B}(\varpi_\ell)$ are isomorphic as P-crystals. Furthermore

Proposition 2.2.11. *[HN06]*

(i) $\mathcal{M}_{I_0}(M_j)$ *consists of* $m_{T;j}$ *for various sequences* T.

(ii) We have the equality of I_0*-crystals*

$$\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1}) = \bigsqcup_{k\in\mathbb{Z}} (\tau_{n+1,-\ell\delta})^k \left(\bigsqcup_{j=0}^{\ell-1} \mathcal{M}_{I_0}(M_j) \right). \tag{2.2}$$

(iii) The map $\sigma : \mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1}) \to \mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ *defined by* $\sigma(m_{T;j}) = m_{T;j+1}$ *is a* P_{cl}*-crystal automorphism equal to* z_ℓ^{-1}.

(iv) The Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ *are described in terms of tableaux: for* $i \neq 0$ *we have* $\tilde{e}_i \cdot m_{T;j} = m_{T';j}$ *or* 0. *Here* T' *is obtained from* T *by replacing* $i+1$ *by* i. *If it is not possible (i.e. when we have both* $i+1$ *and* i *in* T *or when* $i+1$ *does not occur in* T*), then it is zero. Similarly* $\tilde{f}_i \cdot m_{T;j} = m_{T'';j}$ *or* 0, *where* T'' *is given by replacing* i *by* $i+1$. *For the action of* $\tilde{e}_0$, $\tilde{f}_0$, *we have*

$$\tilde{e}_0 \cdot m_{T;j} = \begin{cases} 0 & \text{if } i_1 \neq 1 \text{ or } i_\ell = n+1, \\ m_{(i_2,\dots,i_\ell,n+1);j-1} & \text{if } i_1 = 1 \text{ and } i_\ell \neq n+1, \end{cases}$$

$$\tilde{f}_0 \cdot m_{T;j} = \begin{cases} 0 & \text{if } i_1 = 1 \text{ or } i_\ell \neq n+1, \\ m_{(1,i_1,\dots,i_{\ell-1});j+1} & \text{if } i_1 \neq 1 \text{ and } i_\ell = n+1. \end{cases}$$

Proposition 2.2.12. *There is a bijection given by* Ξ^0 *between* $\mathcal{M}_{I_0}(M_0)$ *and* $\mathcal{M}(V)$, *where* $V = V_0(\Xi^0(M_0))$ *is the fundamental representation of* $\mathcal{U}_q(\hat{sl}_{n+1})'$ *associated to* $Y_{\ell,0}$. *In particular the monomial crystal* $\mathcal{M}_{I_0}(M_0)$ *is* I_0*-closed.*

Proof. By the previous description, $\mathcal{M}_{I_0}(M_0)$ consists of the monomials $m_{T;0}$ for various sequences T. By applying the map Ξ^0, they are sent to the monomials $m_T = \prod_{p=1}^{\ell} \boxed{i_p}_{\ell+1-2p}$ with $T = (1 \leq i_1 < \cdots < i_\ell \leq n+1)$ and where we set $\boxed{1}_p = Y_{1,p}$ and $\boxed{n+1}_p = Y_{n,p+n+1}^{-1}$ for all $p \in \mathbb{Z}$. They are exactly the monomials occurring in the tableaux sum expressions of q-characters of fundamental modules of $\mathcal{U}_q(\hat{sl}_{n+1})'$ (see [Nak03]). So the image of $\mathcal{M}_{I_0}(M_0)$ by Ξ^0 is equal to $\mathcal{M}(V_0(Y_{\ell,0}))$. Further this set is I_0-q-closed by definition, and $\mathcal{M}_{I_0}(M_0)$ is also I_0-closed □

Now let us consider the monomial crystal $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ with $1 \leq \ell \leq n$. We determine in the next proposition when z_ℓ has the particular form of a shift.

Proposition 2.2.13. *The automorphism* z_ℓ *of* $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ *has the special form of a shift* $\tau_{p,\alpha}$ $(p \in \mathbb{Z}, \alpha \in \mathbb{Z}\delta)$ *if and only if* $\ell = 1, n$ *or* $\ell = r+1$. *Moreover, we have* $z_1 = z_n = \tau_{-n-1,\delta}$ *and* $z_{r+1} = \tau_{-2,\delta}$.

Proof. Assume that $\ell \leq r+1$. We have seen that $z_\ell = \sigma^{-1}$. So it suffices to determine when σ is a shift. We have the equality $\sigma^\ell = \tau_{n+1,-\ell\delta}$. Hence if $\ell = 1$, $\sigma = \tau_{n+1,-\delta}$ is a shift. Assume that $\ell = r+1$. In this case, $M_1 = \tau_{2,-\delta}(M_0) = \sigma(M_0)$. As the crystal $\mathcal{M}(M_0)$ is connected and σ and $\tau_{2,-\delta}$ are automorphisms of crystals, we have $\sigma = \tau_{2,-\delta}$. For the other cases, σ is explicitly known and is not a shift. As ψ and shift automorphisms commute, the result follows for $\ell > r+1$. □

2.2.3 Affinized promotion operators and monomial crystals $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$

In this section, we introduce promotion operators for the level 0 extremal fundamental weight crystals. We describe them in the monomial realizations of $\mathcal{B}(\varpi_\ell)$ $(1 \leq \ell \leq n)$.

Let us begin by some definitions and properties about the promotion operators (see [BST10, FOS09, Sch08, Shi02] and references therein for more details). In type A_n, the highest weight crystal $\mathcal{B}_0(\lambda)$ of highest weight $\lambda \in P_0^+$ can be realized by the semi-standard Young tableaux of shape (λ). The weight function wt is defined by the content of tableaux, i.e.

$$\mathrm{wt}(T) := (w_1(T), \ldots, w_{n+1}(T))$$

where $w_i(T)$ is the number of letters i occurring in the tableau T. It can be viewed as an element of P_0 in the following way: set $\epsilon_i = \Lambda_i - \Lambda_{i-1}$ for $2 \leq i \leq n$, $\epsilon_1 = \Lambda_1$ and $\epsilon_{n+1} = -\epsilon_1 - \cdots - \epsilon_n$. In particular, $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $\Lambda_i = \epsilon_1 + \cdots + \epsilon_i$ $(1 \leq i \leq n)$, and we have $P = \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_{n+1}$. Then $\mathrm{wt}(T)$ corresponds to the element

$$w_1(T)\epsilon_1 + \cdots + w_{n+1}(T)\epsilon_{n+1} \in P_0$$

for all Young tableau T.

Definition 2.2.14. *Let $\mathcal{B}_0 = \mathcal{B}_0(\lambda)$ be a highest weight $\mathcal{U}_q(sl_{n+1})$-crystal of highest weight $\lambda \in P_0^+$. A promotion operator* pr *on $\mathcal{B}_0$ is an operator* $\mathrm{pr} : \mathcal{B}_0 \to \mathcal{B}_0$ *such that*

(i) pr *shifts the content: if* $\mathrm{wt}(T) = (w_1, \ldots, w_{n+1})$ *is the content of* $T \in \mathcal{B}_0$*, then* $\mathrm{wt}(\mathrm{pr}(T)) = (w_{n+1}, w_1, \ldots, w_n)$*,*

(ii) promotion operator has order $n+1$: $\mathrm{pr}^{n+1} = \mathrm{id}$*,*

(iii) $\mathrm{pr} \circ \tilde{e}_i = \tilde{e}_{i+1} \circ \mathrm{pr}$ *and* $\mathrm{pr} \circ \tilde{f}_i = \tilde{f}_{i+1} \circ \mathrm{pr}$ *for $i \in \{1, 2, \ldots, n-1\}$.*

Given a promotion operator pr on a highest weight $\mathcal{U}_q(sl_{n+1})$-crystal $\mathcal{B}_0(\lambda)$ $(\lambda \in P_0^+)$, one defines an associated affine P_{cl}-crystal by setting

$$\tilde{e}_0 := \mathrm{pr}^{-1} \circ \tilde{e}_1 \circ \mathrm{pr} \text{ and } \tilde{f}_0 := \mathrm{pr}^{-1} \circ \tilde{f}_1 \circ \mathrm{pr}.$$

We denote the P_{cl}-crystal hence obtained by $\mathcal{B}_0(\lambda)'_{\mathrm{aff}}$.

It was shown in [Shi02] that the $\mathcal{U}_q(sl_{n+1})$-crystal $\mathcal{B}_0(\lambda)$ $(\lambda \in P_0)$ has a unique promotion operator pr when λ is rectangular (i.e. of the form $k\Lambda_\ell$ with $\ell \in I_0$ and $k \in \mathbb{N}^*$), given by the Schützenberger jeu-de-taquin process. Furthermore the affine P_{cl}-crystal $\mathcal{B}_0(k\Lambda_\ell)'_{\mathrm{aff}}$ obtained by using the promotion operator pr is isomorphic to the crystal basis of a Kirillov-Reshetikhin module associated to $\ell \in I_0, k \in \mathbb{N}^*$ (for a special choice of the spectral parameter $a \in \mathbb{C}^*$ - see [KKM+92]).

From the affine P_{cl}-crystal $\mathcal{B}_0(k\Lambda_\ell)'_{\mathrm{aff}}$, let us consider its affinization $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$ (see also [Kas02b]): this is the P-crystal with vertices in

$$\{z^s T | s \in \mathbb{Z}, T \in \mathcal{B}_0(k\Lambda_\ell)'_{\mathrm{aff}}\}$$

such that for all $s \in \mathbb{Z}$ and $T \in \mathcal{B}_0(k\Lambda_\ell)'_{\mathrm{aff}}$,

$$\mathrm{wt}(z^s T) = \mathrm{wt}(T) + s\delta, \ \tilde{e}_i \cdot z^s T = z^{s+\delta_{i,0}}(\tilde{e}_i \cdot T), \ \tilde{f}_i \cdot z^s T \ = \ z^{s-\delta_{i,0}}(\tilde{f}_i \cdot T).$$

Assume in the following that $\ell \leq r+1$ (the case $\ell > r+1$ is studied at the end of this section). We introduce the affinized promotion operator on $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$.

Definition 2.2.15. *Let us consider the crystal of finite type $\mathcal{B}_0(k\Lambda_\ell)$ (with $k \in \mathbb{N}$ and $\ell \leq r+1$), pr its associated promotion operator and $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$ its affinization. The affinized promotion operator on $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$ is the operator $\mathrm{pr}_{\mathrm{aff}} : \mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}} \to \mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$ such that for all $T \in \mathcal{B}_0(k\Lambda_\ell)'_{\mathrm{aff}}$ and $s \in \mathbb{Z}$,*

$$\mathrm{pr}_{\mathrm{aff}}(z^s T) = z^{s-w_{n+1}(T)}\mathrm{pr}(T).$$

One checks easily the following statements.

Lemma 2.2.16. *The affinized promotion operator $\mathrm{pr}_{\mathrm{aff}}$ of $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$ shifts the content. It satisfies*

$$\mathrm{pr}_{\mathrm{aff}} \circ \tilde{e}_i = \tilde{e}_{i+1} \circ \mathrm{pr}_{\mathrm{aff}} \text{ and } \mathrm{pr}_{\mathrm{aff}} \circ \tilde{f}_i = \tilde{f}_{i+1} \circ \mathrm{pr}_{\mathrm{aff}}$$

for $i \in \{0, 1, \ldots, n\}$ (where $\tilde{e}_{n+1}, \tilde{f}_{n+1}$ are understood to be $\tilde{e}_0, \tilde{f}_0$ respectively). It has infinite order, the weight of $\mathrm{pr}_{\mathrm{aff}}^{n+1}$ being $-k\ell\delta$.

Recall that one has defined an automorphism θ of the Dynkin diagram of type $A_n^{(1)}$ corresponding to a rotation such that $\theta(i) = i+1$ ($i \in I$). Then by the above Lemma, $\mathrm{pr}_{\mathrm{aff}}$ is a θ-twisted automorphism of $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$. Furthermore as the P-crystals $\mathcal{B}(\varpi_\ell)$ and $\mathcal{B}_0(\Lambda_\ell)_{\mathrm{aff}}$ are isomorphic (see [Kas02b]), the affinized promotion operator $\mathrm{pr}_{\mathrm{aff}}$ induces a θ-twisted automorphism of the level 0 fundamental extremal weight crystal $\mathcal{B}(\varpi_\ell)$ ($\ell \leq r+1$). We call it promotion operator of $\mathcal{B}(\varpi_\ell)$, also denoted by $\mathrm{pr}_{\mathrm{aff}}$.

We want to describe the promotion operators of the crystals $\mathcal{B}(\varpi_\ell)$ in the monomial realizations when $\ell \leq r+1$. To do that, let $\phi : \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) \to \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ be the map such that

$$\phi\left(\prod Y_{i,l}^{u_{i,l}}\right) = \prod Y_{i+1,l+1}^{u_{i,l}},$$

the terms e^ν being safely omitted in the definition of ϕ by Remark 1.6.8. Denote by

$$\varphi : \mathcal{B}(\varpi_\ell) \simeq \mathcal{B}_0(\Lambda_\ell)_{\mathrm{aff}} \to \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$$

the isomorphism of P-crystals between $\mathcal{B}_0(\Lambda_\ell)_{\mathrm{aff}}$ and $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$. It is explicitly given by

$$\varphi : z^s T \in \mathcal{B}_0(\Lambda_\ell)_{\mathrm{aff}} \mapsto m_{T;-s} \in \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) \ (s \in \mathbb{Z}, T \in \mathcal{B}_0(\Lambda_\ell)).$$

The following result relates the map $\phi : \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) \to \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ to the promotion operator $\mathrm{pr}_{\mathrm{aff}}$ of $\mathcal{B}(\varpi_\ell)$ introduced above.

Proposition 2.2.17. *Assume that $\ell \leq r+1$. The following diagram commutes*

$$\begin{tikzcd} \mathcal{B}(\varpi_\ell) \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}_{\mathrm{aff}}"'] & \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) \arrow[d, "\phi"] \\ \mathcal{B}(\varpi_\ell) \arrow[r, "\varphi"] & \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) \end{tikzcd}$$

Proof. For $1 \leq k \leq n+1$ and $p \in \mathbb{Z}$, we have

$$\begin{aligned} \phi(\boxed{k}_p) &= \phi(Y_{k-1,p+k}^{-1}Y_{k,p+k-1}) = Y_{k,p+k+1}^{-1}Y_{k+1,p+k} \\ &= \begin{cases} \boxed{k+1}_p & \text{if } k \leq n, \\ \boxed{1}_{p+n+1} & \text{if } k = n+1. \end{cases} \end{aligned}$$

Fix $j \in \mathbb{Z}$ and $T = (1 \leq i_1 < i_2 < \cdots < i_\ell \leq n+1)$ a Young tableau of shape (Λ_ℓ). If $i_\ell \neq n+1$, we have

$$\begin{aligned}\phi(m_{T;j}) &= \phi\left(\prod_{p=1}^{j} \boxed{i_p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{i_p}_{\ell+1-2p+2j}\right)\\ &= \prod_{p=1}^{j} \boxed{i_p+1}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{i_p+1}_{\ell+1-2p+2j}\\ &= m_{\mathrm{pr}(T);j} = \varphi(\mathrm{pr}_{\mathrm{aff}}(z^{-j}T)).\end{aligned}$$

Assume that $i_\ell = n+1$. Then

$$\begin{aligned}\phi(m_{T;j}) &= \prod_{p=1}^{j} \boxed{i_p+1}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell-1} \boxed{i_p+1}_{\ell+1-2p+2j} \times \boxed{1}_{n-\ell+2j+2}\\ &= \prod_{p=2}^{j+1} \boxed{i_{p-1}+1}_{n-\ell-2p+2(j+1)+2} \times \prod_{p=j+2}^{\ell} \boxed{i_{p-1}+1}_{\ell+1-2p+2(j+1)} \times \boxed{1}_{n-\ell+2(j+1)}\\ &= m_{\mathrm{pr}(T);j+1} = \phi(z^{-j-1}\mathrm{pr}(T)) = \varphi(\mathrm{pr}_{\mathrm{aff}}(z^{-j}T)).\end{aligned}$$

□

Remark 2.2.18. *It follows that ϕ is a θ-twisted automorphism of $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$, since $\mathcal{B}(\varpi_\ell)$ and $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ are connected and* $\mathrm{pr}_{\mathrm{aff}}$ *is a θ-twisted automorphism.*

The case $\ell \geq r+1$ is similar to the previous one. The affinized promotion operator of $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$ $(k \in \mathbb{N}^*)$ is the operator

$$\tilde{\mathrm{pr}}_{\mathrm{aff}} : \mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}} \longrightarrow \mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$$

such that for all $T \in \mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$ and $s \in \mathbb{Z}$,

$$\tilde{\mathrm{pr}}_{\mathrm{aff}}(z^sT) = z^{s+k-w_{n+1}(T)}\mathrm{pr}(T).$$

Note that the definition of the affinized promotion operator is different to the one when $\ell \leq r+1$. This provides to the automorphism $\iota_{\mathfrak{h}}$ of $\mathfrak{h}$.

Let us consider the map $\psi \circ \phi \circ \psi^{-1} : \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,n+1-\ell}^{-1}) \to \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,n+1-\ell}^{-1})$. It is a θ^{-1}-twisted automorphism of $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,n+1-\ell}^{-1})$ such that

$$\psi \circ \phi \circ \psi^{-1}\left(\prod Y_{i,l}^{u_{i,l}}\right) = \prod Y_{i-1,l+1}^{u_{i,l}}.$$

It can be related to the promotion operator $\tilde{\mathrm{pr}}_{\mathrm{aff}}$ of $\mathcal{B}_0(k\Lambda_\ell)_{\mathrm{aff}}$: one can check that $\psi \circ \phi \circ \psi^{-1} = \tilde{\mathrm{pr}}_{\mathrm{aff}}^{-1}$.

2.2.4 Application of promotion operators to the study of $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$

In this section, we use promotion operators to obtain a new description of $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$, improving results given in [HN06]. Moreover, we determine the ℓ for which the crystals $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ are closed.

Assume first that $\ell \leq r+1$ (where $n = 2r+1$ is still supposed to be odd). Let us begin by the following remarks. The monomials $\phi^j(Y_{\ell,0}Y_{0,\ell}^{-1}) = Y_{\ell+j,j}Y_{j,\ell+j}^{-1}$ will have a particular importance in the construction of extremal fundamental loop weight modules. One can give them in terms of Young tableaux, thanks to the θ-twisted automorphism ϕ of $\mathcal{M}$:

- if j is such that $\ell + j \leq n+1$, $Y_{\ell+j,j}Y_{j,\ell+j}^{-1} \in \mathcal{M}_{I_0}(M_0)$ and is equal to $m_{T;0}$ with $T = (j+1, j+2, \ldots, j+\ell)$,
- if $1 \leq j \leq \ell - 1$, then $Y_{j,n-\ell+j+1}Y_{n-\ell+j+1,n+j+1}^{-1} \in \mathcal{M}_{I_0}(M_j)$ and is equal to $m_{T;j}$ with $T = (1, 2, \ldots, j, n-\ell+j+2, \ldots, n+1)$.

We will have to consider the finite sub-I_j-crystals of $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$

$$\mathcal{M}_{I_j}(Y_{\ell+j,j+k(n+1)}Y_{j,\ell+j+k(n+1)}^{-1})$$

for $j \in I$ and $k \in \mathbb{Z}$: this is the sub-I_j-crystal of $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ generated by the monomial $Y_{\ell+j,j+k(n+1)}Y_{j,\ell+j+k(n+1)}^{-1}$. Note that one of these crystals can be obtained from another one by application of powers of ϕ.

Proposition 2.2.19. *Let $\ell \leq r+1$. We have the equality of sets*

$$\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigcup_{k\in\mathbb{Z}} (\tau_{n+1,-\ell\delta})^k \left(\bigcup_{j=0}^{n} \mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1}) \right).$$

Proof. As $Y_{\ell+j,j}Y_{j,\ell+j}^{-1} \in \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ for all $0 \leq j \leq n$ and $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is connected,

$$\bigcup_{j=0}^{n} \mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1}) \subset \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$$

as sets.

Let us fix $m \in \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$. The monomial m is of the form $m_{T;j}$ with $T = (1 \leq i_1 < i_2 < \ldots < i_\ell \leq n+1)$ and $j \in \mathbb{Z}$. By application of the shift automorphism, one can assume that $0 \leq j \leq \ell - 1$. So we have to show that $m_{T;j} \in \bigcup_{j=0}^{n} \mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1})$. If $j = 0$, we have $m_{T;0} \in \mathcal{M}_{I_0}(Y_{\ell,0}Y_{0,\ell}^{-1})$. Assume that $1 \leq j \leq \ell - 1$ and set $s = i_{j+1} - 1$. Then $T = (i_1 < \cdots < i_j < s+1 < i_{j+2} < \cdots < i_\ell)$ and by application of the Kashiwara operators $\tilde{e}_1, \ldots, \tilde{e}_{s-1}, \tilde{e}_{s+2}, \ldots, \tilde{e}_n$ on $m_{T;j}$, we show that

$$m_{T;j} \in \mathcal{M}_{I_s}(m_{T';j}) \text{ with } T' = (1 < \cdots < j < s+1 < \cdots < s+\ell-j).$$

By applying $\tilde{e}_1, \ldots, \tilde{e}_{j-1}, \tilde{e}_{s+\ell-j+1}, \ldots, \tilde{e}_n$ and $\tilde{e}_0$ on $m_{T';j}$, it is sent on

$$m_{T'';0} \text{ with } T'' = (s+1 < \cdots < s+\ell) \text{ if } s+\ell \leq n+1,$$

and on

$$m_{T'';u} \text{ with } u = s+\ell-n-1, T'' = (1 < \cdots < u < s+1 < \cdots < n+1) \text{ otherwise.}$$

Furthermore $m_{T'';u} = \phi^s(Y_{\ell,0}Y_{0,\ell}^{-1}) = Y_{\ell+s,s}Y_{s,\ell+s}^{-1}$ by the above remark and $m_{T;j}$ is also contained in $\mathcal{M}_{I_s}(Y_{\ell+s,s}Y_{s,\ell+s}^{-1})$. □

Remark 2.2.20. *One of the questions treated in [HN06, Section 4] is to give an explicit description of monomials occurring in $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ ($\ell \leq r+1$). Actually by the shift automorphism and the description given in [HN06], all the monomials in $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ can* *be obtained from the monomials occurring in* $\bigsqcup_{j=0}^{\ell-1}\mathcal{M}_{I_0}(M_j)$ *(see (2.2)). So this description requires to know* $\left[\ell\cdot\binom{n+1}{\ell}\right]$ *monomials to obtain all the other ones. The preceding proposition improves this result. In fact to determine all the vertices of* $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$*, it suffices to know the monomials occurring in the* $I_{\{0,1\}}$*-crystal* $\mathcal{M}_{I_{\{0,1\}}}(Y_{\ell,0}Y_{0,\ell}^{-1})$ *and to apply* ϕ*. Further a monomial* $m_{T';0}\in\mathcal{M}_{I_{\{0,1\}}}(Y_{\ell,0}Y_{0,\ell}^{-1})$ *is such that* T' *has the form* $T'=(1<i_2<\cdots<i_\ell)$*. So by the above proposition, only* $\binom{n}{\ell-1}$ *monomials are sufficient to determine all the vertices of* $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$*.*

The following lemma will be useful in the following.

Lemma 2.2.21. *Assume that* $\ell=1$ *or* $\ell=r+1$ *and set* $p=n+1$ *or* $p=2$ *respectively. We have the equality of* I_j*-crystals* $(0\leq j\leq n)$

$$\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})=\bigsqcup_{k\in\mathbb{Z}}(\tau_{p,-\delta})^k\left(\mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1})\right)\tag{2.3}$$

Proof. By Proposition 2.2.13, the automorphism z_ℓ has the special form of a shift in the considered cases. Further we know that $(\tau_{-p,\delta})^\ell=\tau_{-n-1,\ell\delta}$. Using (2.2), we obtain

$$\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})=\bigsqcup_{k\in\mathbb{Z}}(\tau_{p,-\delta})^{k\ell}\left(\bigsqcup_{j=0}^{\ell-1}(\tau_{p,-\delta})^j\left(\mathcal{M}_{I_0}(M_0)\right)\right).$$

As $\tau_{p,-\delta}$ and ϕ commute, (2.3) follows. □

Similar equalities of crystals can be given for $\ell\geq r+1$, by using the automorphism ψ. Now we are able to determine the $\ell\in I_0$ for which the monomial crystal $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is closed.

Theorem 2.2.22. *The monomial crystal* $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ *is closed if and only if* $\ell=1,r+1$ *or* n*.*

Proof. Let us begin by the case $\ell\leq r+1$. Assume that the crystal $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is q-closed for $2\leq\ell\leq r$. Consider the monomial

$$M_j=Y_{\ell,2j}Y_{0,n-\ell+1+2j}^{-1}Y_{j,\ell+j}^{-1}Y_{j,n-\ell+1+j}\in\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$$

with $j\neq 0$. We have $\Xi_j(M_j)=Y_{j,\ell+j}^{-1}Y_{j,n-\ell+1+j}$. By Definition 2.2.6, there exists a subset

$$\mathcal{S}_{M_j}\subset\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})\cap\left(M_j\cdot\prod_{l\in\mathbb{Z}}A_{j,l}^{\mathbb{Z}}\right)$$

containing M_j such that its image $\Xi_j\left(\mathcal{S}_{M_j}\right)$ by Ξ_j is the set of ℓ-weights of a representation of $\hat{\mathcal{U}}_j$. By the theory of q–characters of $\hat{\mathcal{U}}_j$-modules, we should have

$$Y_{j,\ell+j}^{-1}Y_{j,n-\ell+1+j}\Xi_j\left(A_{j,\ell+j-1}\right)\in\Xi_j\left(\mathcal{S}_{M_j}\right).$$

Furthermore, the map Ξ_j is injective when it is restricted on the set of monomials $M_j\cdot\left(\prod_{l\in\mathbb{Z}}A_{j,l}^{\mathbb{Z}}\right)$: indeed we have for all monomial $M_j\cdot\left(\prod_{l\in\mathbb{Z}}A_{j,l}^{u_{j,l}}\right)$,

$$\Xi_j\left(M_j\cdot\prod_{l\in\mathbb{Z}}A_{j,l}^{u_{j,l}}\right)=\Xi_j(M_j)\cdot\prod_{l\in\mathbb{Z}}\Xi_j(A_{j,l}^{u_{j,l}})=\Xi_j(M_j)\cdot\prod_{l\in\mathbb{Z}}\tau_{\{j\}}(A_{j,l}^{u_{j,l}})$$

where $\tau_{\{j\}}$ is the map defined in [FM01, Definition 3.2] (the second equality is a consequence of [FM01, Lemma 3.5]). The injectivity is a consequence of the injectivity of $\tau_{\{j\}}$ (see [FM01, Lemma 3.3]).

So the monomial

$$m = M_j \cdot A_{j,\ell+j-1} = Y_{\ell,2j} Y_{0,n-\ell+1+2j}^{-1} Y_{j,\ell+j-2} Y_{j-1,\ell+j-1}^{-1} Y_{j+1,\ell+j-1}^{-1} Y_{j,n-\ell+1+j}$$

should occur in $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$. But this is not the case, m being not of the form (2.1). Hence $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$ is not q-closed when $2 \leq \ell \leq r$.

Now assume that $\ell = 1$ or $\ell = r+1$. In this case, $z_\ell = \tau_{-p,\delta}$ with $p = n+1$ or $p = 2$ respectively and by the above lemma

$$\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1}) = \bigsqcup_{k \in \mathbb{Z}} (\tau_{p,-\delta})^k \left(\mathcal{M}_{I_j}(Y_{\ell+j,j} Y_{j,\ell+j}^{-1}) \right)$$

as I_j-crystals ($0 \leq j \leq n$). By Proposition 2.2.12, the finite crystal $\mathcal{M}_{I_0}(M_0)$ is I_0-closed. As the I_j-crystals $\mathcal{M}_{I_j}(Y_{\ell+j,j} Y_{j,\ell+j}^{-1})$ can be obtained from $\mathcal{M}_{I_0}(M_0)$ by application of powers of ϕ, they are also I_j-closed for all $0 \leq j \leq n$. Then by the above equalities $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$ is closed.

Finally, the result follows for all the $\ell \in I_0$ by using the ι-twisted automorphism ψ (which preserves the notion of q–closed monomial set). $\square$

2.3 Extremal fundamental loop weight modules for $\mathcal{U}_q(sl_{n+1}^{tor})$ when $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ is closed

Assume that $n = 2r+1$ ($r \geq 1$) is odd and $\mathcal{M}_\ell = \mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ is closed (it holds if and only if $\ell = 1, r+1$ or n). In this section, we relate the monomial $\mathcal{U}_q(\hat{sl}_{n+1})$-crystals $\mathcal{M}_\ell$ with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$.

In the first subsection, we construct a new infinite family of representations V_ℓ of $\mathcal{U}_q(sl_{n+1}^{tor})$ (Theorem 2.3.1). We call these representations the extremal fundamental loop weight modules. Let us give the outline of this construction: consider the vector space V_ℓ freely generated by the monomials occurring in $\mathcal{M}_\ell$. For all $0 \leq j \leq n$, we define an action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on it, denoted by $V_\ell^{(j)}$, such that

$$V_\ell^{(j)} = \bigoplus_{k \in \mathbb{Z}} V_k^{(j)}$$

where $V_k^{(j)}$ is a subvector space endowed with a structure of a simple ℓ-highest weight $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module. We show that it defines a $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure in this way on V_ℓ, the compatibility between the action of various vertical subalgebras being a consequence of the existence of promotion operators on $\mathcal{M}_\ell$. Furthermore the q–character of V_ℓ is the sum of monomials occurring in $\mathcal{M}_\ell$ with multiplicity one.

In the second subsection, we study these representations: we show that V_ℓ is irreducible and it is an extremal loop weight module, generated by an extremal vector of ℓ-weight $e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1}$. Furthermore explicit formulas are given for the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on V_ℓ. It is remarkable that these formulas are expressed only from the associated monomial crystal and are "universal" in the following sense: the action on all the extremal fundamental loop weight modules V_ℓ is completely determined by these formulas and by the data of the corresponding monomial crystals $\mathcal{M}_\ell$. This sheds new light on the link between monomial crystals and the theory of q–characters already expected in [HN06]. All these sentences hold for the fundamental ℓ-highest weight modules $V_0(Y_{\ell,0})$ of $\mathcal{U}_q(\hat{sl}_{n+1})'$ with the corresponding monomial crystals $\mathcal{M}_0(Y_{\ell,0})$.

In the third subsection, we specialize q at a root of unity ϵ. We obtain new irreducible finite-dimensional representations of the quantum toroidal algebra $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$.

2.3.1 Construction of the extremal fundamental loop weight modules

Let us begin by the main result of this section.

Theorem 2.3.1. *Assume that $n = 2r+1$ is odd and $\ell = 1, r+1$ or n. There exists a thin representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ whose q–character is the sum of all monomials occurring in $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ with multiplicity one. It is denoted by $V_\ell = V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ and called the extremal fundamental loop weight module of ℓ-weight $e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1}$.*

To construct these representations, let us start with results about the fundamental modules $V_0(Y_{\ell,k})$ of $\mathcal{U}_q(\hat{sl}_{n+1})'$ ($n \in \mathbb{N}^*$, $1 \leq \ell \leq n$, $k \in \mathbb{Z}$). As it is said above, it is isomorphic to the fundamental highest weight $\mathcal{U}_q(sl_{n+1})$-module $V_0(\Lambda_\ell)$. So we begin by recalling some well-known facts about $V_0(\Lambda_\ell)$ which will be useful.

Lemma 2.3.2. *All the weight spaces of the fundamental highest weight $\mathcal{U}_q(sl_{n+1})$-module $V_0(\Lambda_\ell)$ $(1 \leq \ell \leq n)$ are of dimension 1. Furthermore the Weyl group of finite type W_0 acts transitively on* $\mathrm{wt}(V_0(\Lambda_\ell))$.

Proof. We give here a proof of these results by considering the associated crystal $\mathcal{B}_0(\Lambda_\ell)$ of semi-standard Young tableaux $T = (1 \leq i_1 < \cdots < i_\ell \leq n+1)$ of shape (Λ_ℓ). The first statement is easy to show. In fact for $T = (1 \leq i_1 < \cdots < i_\ell \leq n+1)$ of shape (Λ_ℓ), we have $\mathrm{wt}(T) = \epsilon_{i_1} + \cdots + \epsilon_{i_\ell}$. These weights are different to each other and for all $\nu \in \mathrm{wt}(V_0(\Lambda_\ell))$,

$$\dim(V_0(\Lambda_\ell)_\nu) = \sharp\mathcal{B}_0(\Lambda_\ell)_\nu = 1.$$

Let us show that W_0 acts transitively on $\mathrm{wt}(V_0(\Lambda_\ell))$. The action of the simple reflections s_i on ϵ_j $(1 \leq i \leq n,\ 1 \leq j \leq n+1)$ is

$$s_i(\epsilon_j) = \begin{cases} \epsilon_{i+1} & \text{if } i = j, \\ \epsilon_i & \text{if } i = j+1, \\ \epsilon_j & \text{if } i \neq j, j+1. \end{cases}$$

Hence, W_0 acts transitively on $\mathrm{wt}(V_0(\Lambda_\ell))$. □

Proposition 2.3.3. *Let $V_0(Y_{\ell,k})$ be a fundamental module of $\mathcal{U}_q(\hat{sl}_{n+1})'$ ($\ell \in I_0, k \in \mathbb{Z}$). Then $V_0(Y_{\ell,k})$ is a thin $\mathcal{U}_q(\hat{sl}_{n+1})'$-module which admits a basis (v_m) indexed by the vertices of the monomial crystal $\mathcal{M}_0(Y_{\ell,k})$, such that for all $m \in \mathcal{M}_0(Y_{\ell,k})$ and $i \in I_0$, v_m is of ℓ-weight m and*

$$x_{i,0}^+ \cdot v_m = v_{\tilde{e}_i \cdot m},\ x_{i,0}^- \cdot v_m = v_{\tilde{f}_i \cdot m}$$

where $v_0 = 0$ by convention.

Proof. It is known that $\mathrm{Res}(V_0(Y_{\ell,k}))$ is the fundamental highest weight $\mathcal{U}_q(sl_{n+1})$-module $V_0(\Lambda_\ell)$. By the preceding Lemma, its weight spaces are all of dimension one. In particular its ℓ-weight spaces are also of dimension one and $V_0(Y_{\ell,k})$ is a thin $\mathcal{U}_q(\hat{sl}_{n+1})'$-module.

Furthermore $\mathrm{Res}(V_0(Y_{\ell,k}))$ is the extremal weight module of extremal weight Λ_ℓ, generated by an extremal vector v of weight Λ_ℓ. Hence, there exists $\{v_w\}_{w\in W_0}$ such that $v_{\mathrm{Id}} = v$ and

$$x^{\pm}_{i,0} \cdot v_w = 0 \text{ and } (x^{\mp}_{i,0})^{(\pm w(\Lambda_\ell)(h_i))} \cdot v_w = v_{s_i(w)} \text{ if } \pm w(\Lambda_\ell)(h_i) \geq 0.$$

By the above lemma for all $\nu \in \mathrm{wt}(\mathrm{Res}(V_0(Y_{\ell,k})))$, there exists w such that $\nu = w(\Lambda_\ell)$. Then the corresponding vector v_w is non zero of weight ν. As all the weight spaces of $V_0(Y_{\ell,k})$ are of dimension one, $\{v_w\}_{w\in W_0}$ generates $V_0(Y_{\ell,k})$ as a vector space. Furthermore for all $w, w' \in W_0$,

$$w(\Lambda_\ell) = w'(\Lambda_\ell) \Leftrightarrow v_w = v_{w'}.$$

In fact, we have (see [Bou68, Ch. V.3.3, Proposition 2])

$$\begin{aligned} w(\Lambda_\ell) = w'(\Lambda_\ell) &\Leftrightarrow w^{-1}w'(\Lambda_\ell) = \Lambda_\ell \Leftrightarrow w^{-1}w' \in \langle s_i, i \in I_0 - \{\ell\}\rangle \\ &\Leftrightarrow w' \in w \cdot \langle s_i, i \in I_0 - \{\ell\}\rangle. \end{aligned}$$

Fix an ℓ-weight $m \in \mathcal{M}(V_0(Y_{\ell,k})) = \mathcal{M}_0(Y_{\ell,k})$. By that we have said above, one can define v_m as the unique vector v_w ($w \in W_0$) such that $w(\Lambda_\ell) = \mathrm{wt}(m)$. Then $\{v_m | m \in \mathcal{M}_0(Y_{\ell,k})\}$ is a basis of $V_0(Y_{\ell,k})$. Furthermore as the weight subspaces and the ℓ-weight subspaces of $V_0(Y_{\ell,k})$ coincide and are of dimension one, v_m is also an ℓ-weight vector of ℓ-weight m for all $m \in \mathcal{M}_0(Y_{\ell,k})$.

We determine the action of $\mathcal{U}_q^h(\hat{sl}_{n+1})$ on this basis. Fix $m \in \mathcal{M}_{I_0}(Y_{\ell,k})$. For all $i \in I_0$, we have $\mathrm{wt}(m)(h_i) = 0, \pm 1$. Assume that $\mathrm{wt}(m)(h_i) = 0$. Then on the one hand $x^{\pm}_{i,0} \cdot v_m = 0$ by definition of the family $\{v_w\}_{w\in W_0}$. And on the other hand $\tilde{e}_i \cdot m = 0$ and $\tilde{f}_i \cdot m = 0$ by the description of the crystal $\mathcal{M}_0(Y_{\ell,k})$ recalled above. Now assume that $\mathrm{wt}(m)(h_i) = \pm 1$. The vector $S_i(v_m) = x^{\mp}_{i,0} \cdot v_m$ is of the form $v_{m'}$ with $m' \in \mathcal{M}_0(Y_{\ell,k})$ such that

$$\mathrm{wt}(m') = s_i(\mathrm{wt}(m)) = \mathrm{wt}(m) \mp \alpha_i.$$

But the description of $\mathcal{M}_0(Y_{\ell,k})$ shows that the unique monomial of weight $\mathrm{wt}(m) \mp \alpha_i$ is $\tilde{f}_i \cdot m$ (resp. $\tilde{e}_i \cdot m$). Hence m' is equal to $\tilde{f}_i \cdot m$ (resp. $\tilde{e}_i \cdot m$). Finally we have shown that for all $i \in I_0$ and $m \in \mathcal{M}_0(Y_{\ell,k})$,

$$x^{+}_{i,0} \cdot v_m = v_{\tilde{e}_i \cdot m} \text{ and } x^{-}_{i,0} \cdot v_m = v_{\tilde{f}_i \cdot m}.$$

□

In particular, the action of $\mathcal{U}_q(\hat{sl}_{n+1})'$ on the fundamental modules $V_0(Y_{\ell,k})$ is determined by the combinatorics of monomial crystals $\mathcal{M}_0(Y_{\ell,k})$: in fact, the action of operators $x^{\pm}_{i,r}$ ($1 \leq i \leq n$, $r \in \mathbb{Z}$) deduces from the action of the $x^{\pm}_{i,0}$ (given by $\mathcal{M}_0(Y_{\ell,k})$) and the action of $h_{i,r}$ (given by the ℓ-weights $m \in \mathcal{M}_0(Y_{\ell,k})$) from (1.1).

Let us begin the construction of extremal fundamental loop weight modules. Assume that $n = 2r+1$ is odd and $\ell \leq r+1$ (the case $\ell > r+1$ is discussed below at Remark 2.3.5). Consider the monomial $\mathcal{U}_q(\hat{sl}_{n+1})$-crystal $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$, supposed to be closed. This is the case if and only if $\ell = 1$ or $\ell = r+1$ (Theorem 2.2.22). Set $p = n+1$ or $p = 2$ respectively.

Denote by $\mathcal{E}$ (resp. $\mathcal{E}_{j,k}$ for $0 \leq j \leq n$ and $k \in \mathbb{Z}$) the set of monomials occurring in $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,\ell}^{-1})$ (resp. in $(\tau_{p,-\delta})^k \left(\mathcal{M}_{I_j}(Y_{\ell+j,j} Y_{j,\ell+j}^{-1})\right) = \mathcal{M}_{I_j}(Y_{\ell+j,j+kp} Y_{j,\ell+j+kp}^{-1})$). By (2.3), one has $\mathcal{E} = \bigsqcup_{k\in\mathbb{Z}} \mathcal{E}_{j,k}$ for all $0 \leq j \leq n$.

Let
$$V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigoplus_{m\in\mathcal{E}} \mathbb{C}v_m \tag{2.4}$$
be the vector space freely generated by elements of $\mathcal{E}$. For all $0 \leq j \leq n$ and $k \in \mathbb{Z}$ set $V_k^{(j)} = \bigoplus_{m\in\mathcal{E}_{j,k}} \mathbb{C}v_m$ the subspace of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ of dimension $\dim(V_0(\Lambda_\ell))$. In particular, we have
$$V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigoplus_{k\in\mathbb{Z}} V_k^{(j)}.$$
This decomposition can be compared to the equalities of crystals (2.3).

We endow the vector space $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ with a structure of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module as follows ($0 \leq j \leq n$): for all $k \in \mathbb{Z}$, let (v_m) be the basis of the $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module $V_0(Y_{\ell,j+kp})^{(j)}$ defined in Proposition 2.3.3, indexed by the set of monomials
$$\Xi^j\left(\mathcal{M}_{I_j}(Y_{\ell+j,j+kp}Y_{j,\ell+j+kp}^{-1})\right) = \left\{\Xi^j(m) | m \in \mathcal{M}_{I_j}(Y_{\ell+j,j+kp}Y_{j,\ell+j+kp}^{-1})\right\}.$$
Let us define an isomorphism of vector spaces between $V_k^{(j)}$ and $V_0(Y_{\ell,j+kp})^{(j)}$ by
$$\begin{array}{ccc} V_k^{(j)} & \longrightarrow & V_0(Y_{\ell,j+kp})^{(j)} \\ v_m & \mapsto & v_{\Xi^j(m)} \end{array}$$
We endow the vector space $V_k^{(j)}$ with a structure of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module by pulling back the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on $V_0(Y_{\ell,j+kp})^{(j)}$. By direct sum, $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module, denoted by $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$.

Proposition 2.3.4. *There exists a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module on $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ such that for all $j \in I$ the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module is isomorphic to $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$. Furthermore the q–character of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is*
$$\chi_q(V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})) = \sum_{m\in\mathcal{E}} m,$$
where $\mathcal{E}$ is the set of the monomials occurring in $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$.

Proof. To define an action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$, we determine the action of the subalgebras $\hat{\mathcal{U}}_i$ for all $i \in I$. For that, let $j \in I$ be such that $j \neq i$. The action of $\hat{\mathcal{U}}_i$ on $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is the restriction of the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$. Furthermore we set for all $h \in \mathfrak{h}$ and $m \in \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$,
$$k_h \cdot v_m = q^{\mathrm{wt}(m)(h)}v_m.$$

The definition of the action of $\hat{\mathcal{U}}_i$ ($i \in I$) is independent of the choice of $j \in I$, $j \neq i$: for $m \in \mathcal{E}$, the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on the vector v_m is determined by the sub-I_j-crystal $\mathcal{M}_{I_j}(m)$ of $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ and by the ℓ-weight $\Xi^j(m)$. So the action of $\hat{\mathcal{U}}_i$ on v_m is determined by the action of $\tilde{e}_i$ and $\tilde{f}_i$ on m and by the ℓ-weight $\Xi_i(m)$ which are independent of the choice of j.

Let us show that this action endows $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module. We fix two indices $i_1, i_2 \in I$ and we check the relations satisfied by $\hat{\mathcal{U}}_{i_1}$ and $\hat{\mathcal{U}}_{i_2}$. The indices i_1 and i_2 are in the same connected subset I_j of the set of vertices of the Dynkin diagram ($j \in I$). By construction, the action of $\hat{\mathcal{U}}_{i_1}$ and $\hat{\mathcal{U}}_{i_2}$ are restrictions of the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$. As $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$ is a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module, the relations between $\hat{\mathcal{U}}_{i_1}$ and $\hat{\mathcal{U}}_{i_2}$ are satisfied and $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is a $\mathcal{U}_q(sl_{n+1}^{tor})$-module.

By construction, the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module obtained from $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ by restriction is isomorphic to $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$ for all $j \in I$. Furthermore the ℓ-weight of v_m is $\Xi_i(m)$ for the action of $\hat{\mathcal{U}}_i$ ($i \in I$). So m is the ℓ-weight of v_m and the q–character of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is the sum of monomials occurring in $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ with multiplicity one. □

Remark 2.3.5. *Let us consider the case $\ell > r+1$. The monomial crystal $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,n+1-\ell}^{-1})$ is closed for $\ell = n$. We show in the same way as above that there exists also a $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{\varpi_n}Y_{n,0}Y_{0,1}^{-1})$ whose q–character is the sum of monomials occurring in $\mathcal{M}(e^{\varpi_n}Y_{n,0}Y_{0,1}^{-1})$ with multiplicity one.*

Actually this $\mathcal{U}_q(sl_{n+1}^{tor})$-module is related to the previous one for $\ell = 1$ as follows. We have defined an automorphism ι of the Dynkin diagram of type $A_n^{(1)}$. It induces an algebra automorphism of $\mathcal{U}_q(sl_{n+1}^{tor})$ we still denote ι, which sends $x_{i,r}^{\pm}, h_{i,m}, k_h$ to $x_{\iota(i),r}^{\pm}, h_{\iota(i),m}, k_{\iota_{\mathfrak{h}}(h)}$ ($i \in I, r \in \mathbb{Z}, m \in \mathbb{Z} - \{0\}, h \in \mathfrak{h}$). Let us denote by $V(e^{\varpi_n}Y_{n,0}Y_{0,1}^{-1})^\iota$ the $\mathcal{U}_q(sl_{n+1}^{tor})$-module obtained from $V(e^{\varpi_n}Y_{n,0}Y_{0,1}^{-1})$ by twisting the action by ι. Then we show easily that $V(e^{\varpi_n}Y_{n,0}Y_{0,1}^{-1})^\iota$ and $V(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})$ are isomorphic.

2.3.2 Study of the extremal fundamental loop weight modules

In this section, we study the $\mathcal{U}_q(sl_{n+1}^{tor})$-modules $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$, where $n = 2r+1$ is supposed to be odd and $\ell = 1, n$ or $\ell = r+1$. We set $p = n+1$ or $p = 2$ respectively.

Proposition 2.3.6. *The $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is integrable. Moreover, it satisfies properties (iii) and (iv) of Remark 1.3.3 with weight subspaces of dimension one.*

Proof. Assume first that $\ell = 1, r+1$. The q–character of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is known: this is the sum of monomials occurring in $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ with multiplicity one. Furthermore one has the equality of I_0-crystals

$$\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigsqcup_{k\in\mathbb{Z}} (\tau_{p,-\delta})^k \left(\mathcal{M}_{I_0}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})\right).$$

For all $m \in \mathcal{M}_{I_0}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ and $k \in \mathbb{Z}$, $\mathrm{wt}((\tau_{p,-\delta})^k(m)) = \mathrm{wt}(m) - k\delta$. So to prove that the weight spaces of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ are of dimension one, we have to show that the weights of monomials occurring in $\mathcal{M}_{I_0}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ are different to each other. More precisely, it is sufficient to show that the sum

$$\sum_{m\in\mathcal{M}_{I_0}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})} e\left(\mathrm{wt}(\Xi^0(m))\right) \in \bigoplus_{\nu\in P_0} \mathbb{Z}e(\nu)$$

is without multiplicity. This follows from the above results: it is the character of the $\mathcal{U}_q(sl_{n+1})$-module $V_0(\Lambda_\ell)$.

For all $j \in I$, the representation $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is completely reducible as a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module and we have

$$V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)} = \bigoplus_{p\in\mathbb{Z}} V_0(Y_{\ell,j+kp})^{(j)}. \tag{2.5}$$

As the representations $V_0(Y_{\ell,j+kp})$ are all integrable, it holds for $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$. Furthermore $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ satisfies the stronger property (iv) of Remark 1.3.3: in fact the representations $V_0(Y_{\ell,j+kp})$ are all isomorphic as $\mathcal{U}_q(sl_{n+1})$-modules and satisfy property (iv). Hence we have $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})_{\nu+N\alpha_i} = \{0\}$ for all $\nu \in P$, $i \in I$, $N >> 0$.

Finally, the case $\ell = n$ is deduced from the case $\ell = 1$ by the ι-twisted automorphism ψ. □

Theorem 2.3.7. *The $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is an extremal loop weight module generated by the vector $v_{e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1}}$ of ℓ-weight $e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1}$.*

Proof. We treat the case $\ell = 1, r+1$ (the case $\ell = n$ can be deduced from $\ell = 1$ by using ψ). The formulas (2.5) imply immediately the third point of Definition 1.7.1. The first two points are consequences of the following lemmas. □

Lemma 2.3.8. *Let $\mathcal{M}'$ be a sub-$\mathcal{U}_q(\hat{sl}_{n+1})$-crystal of $\mathcal{M}$. Assume that V is a $\mathcal{U}_q(\hat{sl}_{n+1})$-module with basis $(v_m)_{m\in\mathcal{M}'}$ satisfying*

$$\mathrm{wt}(v_m) = \mathrm{wt}(m),\ (x_i^+)^{(k)} \cdot v_m = v_{\tilde{e}_i^k\cdot m} \text{ and } (x_i^-)^{(k)} \cdot v_m = v_{\tilde{f}_i^k\cdot m} \tag{2.6}$$

for all $m \in \mathcal{M}', i \in I$ and $k \in \mathbb{N}$, where $v_0 = 0$ by convention. If the monomial m is extremal of weight λ, then the vector v_m is an extremal vector of weight λ. Furthermore if the crystal $\mathcal{M}'$ is connected, then the $\mathcal{U}_q(\hat{sl}_{n+1})$-module V is cyclic generated by any v_m with $m \in \mathcal{M}'$.

Proof. Assume that m is extremal of weight λ: there exists $\{m_w\}_{w\in W}$ such that $m_{Id} = m$ and

$$\begin{aligned} &\tilde{e}_i \cdot m_w = 0 \text{ and } (\tilde{f}_i)^{w(\lambda)(h_i)} \cdot m_w = m_{s_i(w)} \text{ if } w(\lambda)(h_i) \geq 0,\\ &\tilde{f}_i \cdot m_w = 0 \text{ and } (\tilde{e}_i)^{-w(\lambda)(h_i)} \cdot m_w = m_{s_i(w)} \text{ if } w(\lambda)(h_i) \leq 0. \end{aligned} \tag{2.7}$$

For all $w \in W$, set $v_w = v_{m_w}$. By (2.6) and (3.8), $\{v_w\}_{w\in W}$ satisfies $v_{Id} = v_m$ and

$$x_i^{\pm} \cdot v_w = 0 \text{ if } \pm w(\lambda)(h_i) \geq 0 \text{ and } (x_i^{\mp})^{(\pm w(\lambda)(h_i))} \cdot v_w = v_{s_i(w)}.$$

Hence the vector v_m is extremal of weight λ.

Assume that the crystal $\mathcal{M}'$ is connected and fix $m \in \mathcal{M}'$. For $m' \in \mathcal{M}'$, there exists a product s of Kashiwara operators such that $s(m) = m'$. Consider the corresponding operator $S \in \mathcal{U}_q(\hat{sl}_{n+1})$ at the level of V, i.e. S has the same expression as s where the operators $\tilde{e}_i^k$ (resp. $\tilde{f}_i^k$) are replaced by $(x_i^+)^{(k)}$ (resp. $(x_i^-)^{(k)}$) in the product ($k \in \mathbb{N}, i \in I$). By (2.6), $S(v_m) = v_{s(m)} = v_{m'}$ and the $\mathcal{U}_q(\hat{sl}_{n+1})$-module V is cyclic generated by v_m. □

Lemma 2.3.9. *Assume that $\ell = 1, r+1$. The ℓ-weight vector $v_{e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}} \in V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is an extremal vector of weight ϖ_ℓ for the action of $\mathcal{U}_q^h(sl_{n+1}^{tor})$. Furthermore*

$$V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}) = \mathcal{U}_q^h(sl_{n+1}^{tor}) \cdot v_{e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}}.$$

Proof. Let us begin to show that the basis (v_m) of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ introduced in (2.4) satisfies properties (2.6). For all $m \in \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$, v_m is an ℓ-weight vector of ℓ-weight m and $\mathrm{wt}(v_m) = \mathrm{wt}(m)$. Fix $i \in I$ and let $j \in I$ be such that $j \neq i$. As a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module, $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ is completely reducible (see (2.5)) and there exists $k \in \mathbb{Z}$ such that $v_m \in V_0(Y_{\ell,j+kp})^{(j)}$. As properties (2.6) are satisfied in $V_0(Y_{\ell,j+kp})^{(j)}$ (Proposition 2.3.3), it holds on v_m for $i \in I$.

From there the result is a direct consequence of the above lemma and the fact that $e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}$ is extremal in the connected crystal $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ (Theorem 2.2.4). □

Proposition 2.3.10. *The $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is irreducible. Furthermore it is simple as a $\mathcal{U}_q^h(sl_{n+1}^{tor})$-module and* $\mathrm{Res}(V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1}))$ *is isomorphic to $V(\varpi_\ell)$.*

Proof. Let V be a non trivial sub-$\mathcal{U}_q^h(sl_{n+1}^{tor})$-module of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$. As the weight spaces of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ are all of dimension one, there exists $m \in \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ such that $v_m \in V$. By Lemma 2.3.8, v_m generates $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ and $V = V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$. Hence $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is simple as a $\mathcal{U}_q^h(sl_{n+1}^{tor})$-module and as a $\mathcal{U}_q(sl_{n+1}^{tor})$-module. Furthermore, $\mathrm{Res}(V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1}))$ is an integrable $\mathcal{U}_q(\hat{sl}_{n+1})$-module generated by the extremal vector $v_{e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1}}$ of weight ϖ_ℓ. Then by Theorem 1.4.7, $\mathrm{Res}(V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1}))$ is isomorphic to $V(\varpi_\ell)$. $\square$

Some readers may expect from this result that the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ can be obtained from the extremal weight module $V(\varpi_\ell)$ by an evaluation morphism, but this is not the case for the following reasons (which generalize arguments given in [Her09]): $\mathrm{Res}(V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1}))$ is isomorphic to the $\mathcal{U}_q(\hat{sl}_{n+1})$-module $V(\varpi_\ell)$. In particular,

$$\tau_{p,-\delta} : V(\varpi_\ell) \to V(\varpi_\ell), v_m \mapsto v_{\tau_{p,-\delta}(m)} \text{ for all } m \in \mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$$

is a $\mathcal{U}_q(\hat{sl}_{n+1})'$-automorphism of $V(\varpi_\ell)$ (with $p = n+1$ or $p = 2$ if $\ell = 1, n$ or $\ell = r+1$ respectively). If $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is obtained from an evaluation morphism $\mathcal{U}_q(sl_{n+1}^{tor}) \to \mathcal{U}_q^h(sl_{n+1}^{tor})$, $\tau_{p,-\delta}$ should induce an automorphism of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$. But it does not commute with the action of the $x_{i,r}^{\pm}, h_{i,r}$ for $i \in I$ and $r \in \mathbb{Z}-\{0\}$. In the same way, $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ can not be obtained from an evaluation morphism $\mathcal{U}_q(sl_{n+1}^{tor}) \to \mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ $(j \in I)$. In fact, $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is completely reducible as a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module and is a direct sum of fundamental modules (see (2.5)). But it is a simple $\mathcal{U}_q(sl_{n+1}^{tor})$-module.

Remark 2.3.11. *Let us denote $\mathcal{U}_q(sl_{n+1}^{tor})'$ the quantum toroidal algebra without derivation element, i.e. this is the subalgebra of $\mathcal{U}_q(sl_{n+1}^{tor})$ generated by $x_{i,r}^{\pm}$ $(i \in I, r \in \mathbb{Z}), h_{i,m}$ $(i \in I, m \in \mathbb{Z}-\{0\})$ and k_h $(h \in \sum \mathbb{Q}h_i)$. An automorphism Ψ of $\mathcal{U}_q(sl_{n+1}^{tor})'$ which exchanges vertical and horizontal quantum affine subalgebras is defined in [Mik99]. Denote by $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})^\Psi$ the $\mathcal{U}_q(sl_{n+1}^{tor})'$-module obtained from $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ by twisting the action by Ψ. It would be interesting to determine if $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})^\Psi$ is already known, for example if it is of ℓ-highest weight. Actually this is not the case: for the vertical quantum affine subalgebra $\mathcal{U}_q^v(sl_{n+1}^{tor})'$, it is an integrable and cyclic module which is reducible. Further as a $\mathcal{U}_q^h(sl_{n+1}^{tor})'$-module, it is completely reducible, direct sum of irreducible finite-dimensional representations. So $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})^\Psi$ cannot be an ℓ-highest weight module or an ℓ-lowest weight module.*

From now on, let $\mathcal{M}_0'$ be a subcrystal of $\mathcal{M}_0$ over $\mathcal{U}_q(sl_{n+1})$ (resp. $\mathcal{M}'$ subcrystal of $\mathcal{M}$ over $\mathcal{U}_q(\hat{sl}_{n+1})$). Let us consider the vector space V with basis (v_m) indexed by the vertices of $\mathcal{M}_0'$ (resp. $\mathcal{M}'$). We define an action of $\mathcal{U}_q(\hat{sl}_{n+1})'$ (resp. $\mathcal{U}_q(sl_{n+1}^{tor})$) on V by the following formulas

$$\begin{aligned}
x_{i,r}^{+} \cdot v_m &= q^{r(p_i(m)-1)} v_{\tilde{e_i}\cdot m},\\
x_{i,r}^{-} \cdot v_m &= q^{r(q_i(m)+1)} v_{\tilde{f_i}\cdot m},\\
\phi_{i,\pm s}^{\pm} \cdot v_m &= \pm(q-q^{-1})\left(\varphi_i(m) q^{\pm s(q_i(m)+1)} - \varepsilon_i(m) q^{\pm s(p_i(m)-1)}\right) v_m,\\
k_h \cdot v_m &= q^{\mathrm{wt}(m)(h)} v_m.
\end{aligned} \tag{2.8}$$

with $r \in \mathbb{Z}$, $s > 0$, $i \in I_0$ (resp. $i \in I$) and $h \in \mathfrak{h}_0$ (resp. $h \in \mathfrak{h}$), and where $v_0 = 0$ by convention. Note that $p_i(m)$ is well defined only if $\varepsilon_i(m) > 0$ or equivalently if $\tilde{e_i} \cdot m \neq 0$

and $q_i(m)$ is well defined only if $\varphi_i(m) > 0$ or equivalently if $\tilde{f}_i \cdot m \neq 0$. Then, these expressions make sense.

Theorem 2.3.12. *(i) Set $n \in \mathbb{N}^*$, $1 \leq \ell \leq n$. Assume that $\mathcal{M}'_0 = \mathcal{M}_0(Y_{\ell,k})$. Then formulas (2.8) endow V with a structure of $\mathcal{U}_q(\hat{sl}_{n+1})'$-module isomorphic to the fundamental module $V_0(Y_{\ell,k})$.*

(ii) Assume that $n = 2r+1$ is odd and $\ell = 1, r+1, n$. Set $\mathcal{M}' = \mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$. Then formulas (2.8) endow V with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module isomorphic to the extremal fundamental loop weight module $V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$.

Proof. The action of the horizontal quantum affine subalgebra and the action of the Cartan subalgebra are known on the basis $(v_m)_{m \in \mathcal{M}'}$ for the $\mathcal{U}_q(\hat{sl}_{n+1})'$-module $V_0(Y_{\ell,k})$ and for the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$. From (1.1) it is straightforward to deduce the action of the $x_{i,r}^{\pm}$ on these modules ($r \in \mathbb{Z}$). We obtain formulas (2.8) given only from the corresponding monomial crystal. □

Remark 2.3.13. *In [Her11], the algebra $\mathcal{U}_q(\hat{sl}_\infty)$ is introduced as the quantum affinization of $\mathcal{U}_q(sl_\infty)$. It is defined by the same generators and relations as in Definition 1.5.1 with the infinite Cartan matrix $C = (C_{i,j})_{i,j \in \mathbb{Z}}$ such that*

$$C_{i,i} = 2,\ C_{i,i+1} = -1,\ C_{i+1,i} = -1, C_{i,j} = 0$$

if $i-j \notin \{-1,0,1\}$. The representation theory of $\mathcal{U}_q(\hat{sl}_\infty)$ is similar to the one of $\mathcal{U}_q(sl_{n+1}^{tor})$: the simple ℓ-highest weight modules are parametrized by Drinfeld polynomials. In particular, the fundamental modules can be defined and they are the inductive limit of the fundamental modules for the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})'$ when $n \to \infty$ (see [Her11, Theorem 3.8] and [Her11, Proposition 3.11]). So, the previous results about the fundamental modules of $\mathcal{U}_q(\hat{sl}_{n+1})'$ extend directly to the case of the fundamental modules of $\mathcal{U}_q(\hat{sl}_\infty)$.

Remark 2.3.14. *As we have said, relations between monomial crystals and the set of monomials occurring in the q–character of representations are known and have combinatorial origin (see [Her11, HN06, Nak03]). The above results, in particular Theorem 2.3.12, give one way to better understand the representation theoretical meaning of this narrow link expected in [HN06]. In fact, formulas (2.8) hold for all the fundamental $\mathcal{U}_q(\hat{sl}_{n+1})'$-modules $V_0(Y_{\ell,k})$ and for all the extremal fundamental loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-modules $V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$. Hence the knowledge of these representations is reduced to the one of the corresponding crystals $\mathcal{M}_0(Y_{\ell,k})$ and $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ respectively, which is totally combinatorial.*

Example 2.3.15. *Assume that $n = 3$ and $\ell = 1$. We study the extremal fundamental loop weight module $V(e^{\varpi_1} Y_{1,0} Y_{0,1}^{-1})$ for $\mathcal{U}_q(sl_4^{tor})$. Let us consider the monomial crystal $\mathcal{M}(e^{\varpi_1} Y_{1,0} Y_{0,1}^{-1})$. It is closed and $p = 4$ in this case. Using the notation introduced above, $\mathcal{E} = \sqcup_{k \in \mathbb{Z}} \mathcal{E}_{0,k}$ and we have*

$$\mathcal{E}_{0,0} = \left\{ e^{\varpi_1} Y_{1,0} Y_{0,1}^{-1}, Y_{2,1} Y_{1,2}^{-1}, Y_{3,2} Y_{2,3}^{-1}, Y_{0,3} Y_{3,4}^{-1} \right\}.$$

$\mathcal{E}_{0,k}$ can be obtained from $\mathcal{E}_{0,0}$ by applying $\tau_{4k,-\delta}$. In the same way, we obtain $\mathcal{E}_{j,k}$ by applying ϕ^{j+4k} to $\mathcal{E}_{0,0}$. Then the q–character of the extremal fundamental loop weight

module $V(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})$ *is*

$$\chi_q(V(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})) = \sum_{k\in\mathbb{Z}} e^{\varpi_1 - k\delta} Y_{1,4k}Y_{0,1+4k}^{-1} + Y_{2,1+4k}Y_{1,2+4k}^{-1} + Y_{3,2+4k}Y_{2,3+4k}^{-1} + Y_{0,3+4k}Y_{3,4+4k}^{-1}.$$

Furthermore the action is explicitly given by the crystal $\mathcal{M}(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})$ *and by the formulas (2.8). This module is already constructed in [Her09].*

Remark 2.3.16. *After this paper appeared on the arXiv, the constructions in [FJMM13] were brought to our attention by H. Nakajima: some representations over the d-deformation* $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ *of the quantum toroidal algebra are obtained as quantum version of a module over a Lie algebra of difference operators. They are called vector representations in [FJMM13]. Our works give another way to define these representations. Actually let* $\omega : \mathcal{U}_q(sl_{n+1}^{tor}) \to \mathcal{U}_{q^{-1}}(sl_{n+1}^{tor})$ *be the map sending* $x_{i,r}^{\pm}, h_{i,m}, k_h$ *to* $x_{i,r}^{\mp}, h_{i,m}, k_h$ $(i \in I, r \in \mathbb{Z}, m \in \mathbb{Z} - \{0\}, h \in \mathfrak{h})$. ω *extends to an isomorphism of algebras. For* $u \in \mathbb{C}^*$, *let* $[V(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})]_u$ *be the* $\mathcal{U}_q(sl_{n+1}^{tor})$*-module obtained from* $V(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})$ *by twisting the action by* $t_{uq^{-1}} \circ \omega$. *Then* $[V(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})]_u$ *is isomorphic to a vector representation where we specialized the parameter* d *at* 1 *(this representation is denoted by* $V^{(2)}(u)$ *in [FJMM13]).*

Example 2.3.17. *Assume that* $n = 3$ *and* $\ell = 2$. *Let us study the extremal fundamental loop weight module* $V(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})$ *of* $\mathcal{U}_q(sl_4^{tor})$. *Consider the closed monomial crystal* $\mathcal{M}(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})$. *In this case,* $p = 2$ *and we have*

$$\mathcal{E}_{0,0} = \left\{ \begin{array}{c} e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1}, Y_{1,1}Y_{2,2}^{-1}Y_{3,1}Y_{0,2}^{-1}, Y_{1,1}Y_{3,3}^{-1}, \\ Y_{3,1}Y_{1,3}^{-1}, Y_{1,3}^{-1}Y_{2,2}Y_{3,3}^{-1}Y_{0,2}, Y_{2,4}^{-1}Y_{0,2} \end{array} \right\}.$$

To describe all the monomials occurring in $\mathcal{M}(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})$, *it is sufficient to consider only the sub-*$I_{\{0,1\}}$*-crystal*

$$Y_{2,0}Y_{0,2}^{-1} \xrightarrow{2} Y_{1,1}Y_{2,2}^{-1}Y_{3,1}Y_{0,2}^{-1} \xrightarrow{3} Y_{1,1}Y_{3,3}^{-1}$$

and to apply the θ*-twisted automorphism* ϕ *(Remark 2.2.20). The* q*–character of* $V(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})$ *is*

$$\begin{aligned}\chi_q(V(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})) = \sum_{k\in\mathbb{Z}} e^{\varpi_2 - k\delta} & Y_{2,2k}Y_{0,2+2k}^{-1} + Y_{1,1+2k}Y_{2,2+2k}^{-1} + Y_{3,1+2k}Y_{0,2+2k}^{-1} \\ & + Y_{1,3+2k}^{-1}Y_{3,1+2k} + Y_{3,1+2k}Y_{1,3+2k}^{-1} + Y_{1,3+2k}^{-1}Y_{2,2+2k} \\ & + Y_{3,3+2k}^{-1}Y_{0,2+2k} + Y_{2,4+2k}^{-1}Y_{0,2+2k},\end{aligned}$$

and the action of $\mathcal{U}_q(sl_4^{tor})$ *on* $V(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})$ *is explicitly given by the crystal* $\mathcal{M}(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})$ *and formulas (2.8).*

2.3.3 Finite-dimensional representations at roots of unity

The existence of shift automorphisms for $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,d_\ell}^{-1})$ is related to finite-dimensional representations of quantum toroidal algebras at roots of unity. We explain that in this section.

So assume that $n = 2r+1$ is odd ($r \geq 1$) and ($\ell = 1, n$) or ($\ell = r+1$). In this case $\mathcal{M}(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ is closed and its automorphism z_ℓ has the special form of a shift $\tau_{-p,\delta}$ with $p = n+1$ or $p = 2$ respectively.

Set $L \geq 1$ and ϵ a primitive (pL)-root of unity (we assume also that $p \neq 2$ or $L > 1$ in the following). Let $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$ be the algebra defined as $\mathcal{U}_q(sl_{n+1}^{tor})$ with ϵ instead of q (without divided powers and derivation element).

For $N \in \mathbb{N}^*$, let $\Gamma_N : \mathbb{Z}[Y_{i,l}^{\pm 1}]_{i \in I, l \in \mathbb{Z}} \to \mathbb{Z}[Y_{i,\bar{l}}^{\pm 1}]_{i \in I, \bar{l} \in \mathbb{Z}/N\mathbb{Z}}$ be the map defined by sending the variables $Y_{i,l}^{\pm 1}$ to $Y_{i,\bar{l}}^{\pm 1}$ ($i \in I, l \in \mathbb{Z}$). Set $\mathcal{S}_\epsilon$ the image of a monomial set $\mathcal{S}$ by $\Gamma_{(pL)}$.

Consider the monomial set $\mathcal{E}_\epsilon$. By the existence of the shift automorphism $\tau_{-p,\delta}$, we have

$$\mathcal{E}_\epsilon = \bigsqcup_{0 \leq k \leq L-1} (\tau_{p,-\delta})^k \left((\mathcal{E}_{j,k})_\epsilon \right)$$

with $j \in I$. One checks easily that $\mathcal{E}_\epsilon$ is closed.

By specializing the representations $V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})$ at a root of unity ϵ, we obtain

Theorem 2.3.18. *Assume that ϵ is a primitive (pL)-root of unity. There is an irreducible $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$-module $V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})_\epsilon$ of dimension $L \times \begin{pmatrix} n+1 \\ \ell \end{pmatrix}$ such that*

$$\chi_\epsilon(V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})_\epsilon) = \sum_{m \in \mathcal{E}_\epsilon} m.$$

Furthermore there exists a basis (v_m) of $V(e^{\varpi_\ell} Y_{\ell,0} Y_{0,d_\ell}^{-1})_\epsilon$ indexed by $\mathcal{E}_\epsilon$ such that the action on it is given by

$$\begin{array}{rcl} x_{i,r}^{+} \cdot v_m & = & \epsilon^{r(p_i(m)-1)} v_{\tilde{e}_i \cdot m}, \\ x_{i,r}^{-} \cdot v_m & = & \epsilon^{r(q_i(m)+1)} v_{\tilde{f}_i \cdot m}, \\ \phi_{i,\pm s}^{\pm} \cdot v_m & = & \pm(\epsilon - \epsilon^{-1}) \left(\varphi_i(m) \epsilon^{\pm s(q_i(m)+1)} - \varepsilon_i(m) \epsilon^{\pm s(p_i(m)-1)} \right) v_m, \\ k_i^{\pm} \cdot v_m & = & \epsilon^{\pm(\varphi_i(m) - \varepsilon_i(m))} v_m. \end{array}$$

2.4 Extremal loop weight modules for non closed monomial crystals

In this section, we still assume that $n = 2r+1$ is odd and we discuss the case where the considered monomial crystal $\mathcal{M}'$ is not closed. It is not possible here to construct an integrable module whose q–character is a sum of monomials occurring in $\mathcal{M}'$. In fact some monomials miss and we have to consider a larger closed monomial crystal $\overline{\mathcal{M}'}$ containing it. It is obtained from $\mathcal{M}'$ by adding other monomial crystals. But its structure is more complicated than $\mathcal{M}'$ and it is difficult for us to construct systematically a possible representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ associated to $\overline{\mathcal{M}'}$.

So in this section, we propose to treat an example of such a construction. Assume in the following that $n = 3$ and consider the crystal $\mathcal{M}(e^{2\varpi_1} Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1})$ which is not closed. We determine a closed monomial crystal $\overline{\mathcal{M}}(e^{2\varpi_1} Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1})$ containing it and we construct a representation $V(e^{2\varpi_1} Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1})$ of $\mathcal{U}_q(sl_4^{tor})$ such that its q–character is the sum of monomials occurring in $\overline{\mathcal{M}}(e^{2\varpi_1} Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1})$ with multiplicity one (Theorem 2.4.6). Furthermore we will see that $V(e^{2\varpi_1} Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1})$ satisfies the definition of extremal loop weight module.

In the first subsection, we study the crystal $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and we determine a closed monomial crystal $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ containing it.

The construction of the $\mathcal{U}_q(sl_4^{tor})$-module $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is done in the second subsection. The process is the same as in the closed case: we consider the vector space freely generated by the vertices m of $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and we define an action of $\mathcal{U}_q(sl_4^{tor})$ by pasting together some finite-dimensional representations of the vertical quantum affine subalgebras $\mathcal{U}_q^{v,j}(sl_4^{tor})$ $(j \in I)$.

In the third subsection, we study the representation $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$: it is an integrable representation of $\mathcal{U}_q(sl_4^{tor})$ which is thin and irreducible. Furthermore $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is an extremal loop weight module of ℓ-weight $e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}$.

In the last subsection, we specialize q at roots of unity ϵ. We get finite-dimensional representations of the quantum toroidal algebra $\mathcal{U}_\epsilon(sl_4^{tor})'$.

Remark 2.4.1. *It could be interesting to construct other extremal fundamental loop weight modules of ℓ-weight $e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1}$ with $2 \leq \ell \leq r$ in the same way. The first crystal $\mathcal{M}(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ which is not closed is obtained for $n = 5$ and $\ell = 2$. We are led to consider the following closed crystal*

$$\overline{\mathcal{M}}(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1}) = \mathcal{M}(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1}) \oplus \bigoplus_{s\in\mathbb{N}^*} \mathcal{M}(e^{2\Lambda_1 - s\delta}Y_{1,1}Y_{1,-1+6s}Y_{0,2}^{-1}Y_{0,6s}^{-1}).$$

which contains $\mathcal{M}(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})$. The maps ϕ and $\tau_{6,-2\delta}$ are automorphisms of it and the P_{cl}-crystals $\mathcal{M}(e^{\varpi_2}Y_{2,0}Y_{0,2}^{-1})/(\tau_{6,-2\delta})$ and $\mathcal{M}(e^{2\Lambda_1-s\delta}Y_{1,1}Y_{1,-1+6s}Y_{0,2}^{-1}Y_{0,6s}^{-1})/(\tau_{6,-2\delta})$ have 30 vertices and 36 vertices respectively.

The example we propose to treat in this section is simpler than the case of the extremal fundamental loop weight modules and we focus only on this situation for the sake of clarity and simplicity.

2.4.1 Study of the $\mathcal{U}_q(\hat{sl}_4)$-crystal $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$

We refer to the Appendix for explicit descriptions of all the crystals considered in this section (see also [Man12b]). Let us study the monomial crystal $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$: the maps ϕ and $\tau_{4,-2\delta}$ are automorphisms of $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Furthermore straightforward computations lead to the following result.

Proposition 2.4.2. *(i) We have the equality of sets*

$$\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigcup_{k\in\mathbb{Z}} (\tau_{4,-2\delta})^k \left(\bigcup_{j=0}^{3} \mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1}) \right).$$

(ii) For all $j \in I$, the monomial crystal $\mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1})$ is I_j-q-closed. More precisely, we have the bijection of monomial sets

$$\Xi^j : \mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1}) \longrightarrow \mathcal{M}(V_0(Y_{1,1+j}Y_{1,-1+j})^{(j)})$$

where $V_0(Y_{1,1+j}Y_{1,-1+j})$ is the simple ℓ-highest weight representation of $\mathcal{U}_q(\hat{sl}_4)'$ of ℓ-highest weight $Y_{1,1+j}Y_{1,-1+j}$.

(iii) *For all $j \in I$, the I_j-crystal $\mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1})$ is not q-closed: the monomial $\phi^j(Y_{1,-1}Y_{3,5}^{-1}Y_{0,0}^{-1}Y_{0,4})$ occurs in this crystal, but it is not the case of $\phi^j(Y_{1,-1}Y_{3,5}^{-1}Y_{0,0}^{-1}Y_{0,4}\cdot A_{0,-1}) = \phi^j(Y_{1,5}Y_{1,-1}Y_{0,6}^{-1}Y_{0,0}^{-1})$.*

Hence, we are led to consider the crystal $\mathcal{M}(e^{2\varpi_1+\delta}Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1})$ which is also not closed. More generally we have to deal with all the monomial crystals $\mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ with $s \in \mathbb{N}$. We set

$$\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{s\in\mathbb{N}} \mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}).$$

For all $(k,s) \in \mathbb{Z}\times\mathbb{N}$ and $j \in I$, denote by

– $\mathcal{M}_{j,k,s}^1$ the sub-I_j-crystal of $\mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ generated by the monomial $\phi^{j+4k}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$,

– $\mathcal{M}_{j,k,s}^2$ the sub-I_j-crystal of $\mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ generated by the monomial $\phi^{j+4k}(Y_{1,1}Y_{1,-3-4s}^{-1}Y_{2,-4-4s}Y_{0,2}^{-1})$.

Proposition 2.4.3. *(i) For all $s \in \mathbb{N}$ and $j \in I$, one has the equality of I_j-crystals*

$$\mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}) = \bigoplus_{k\in\mathbb{Z}}\left(\mathcal{M}_{j,k,s}^1 \oplus \mathcal{M}_{j,k,s}^2\right).$$

(ii) *For all $j \in I$, $k \in \mathbb{Z}$ and $s \geq 1$, the monomial crystal $\mathcal{M}_{j,k,s} = \mathcal{M}_{j,k,s}^1 \oplus \mathcal{M}_{j,k,s-1}^2$ is I_j-q-closed. More precisely, we have the bijection of monomial sets*

$$\Xi^j : \mathcal{M}_{j,k,s} \longrightarrow \mathcal{M}\left(V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}\right)$$

where $V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})$ is the ℓ-highest weight representation of $\mathcal{U}_q(\hat{sl}_4)'$ of ℓ-highest weight $Y_{1,1+j+4k}Y_{1,-1+j+4k-4s}$.

The proof of these statements is straightforward. As a consequence of these results, we have

Corollary 2.4.4. *The monomial crystal $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is closed.*

Proposition 2.4.5. *$\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is a monomial realization of the P-crystal $\mathcal{B}(2\varpi_1)$. Furthermore, the monomials $M_s = e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}$ are extremal of weight $2\varpi_1 + s\delta$ ($s \in \mathbb{N}$).*

Proof. The monomial crystal $\mathcal{M}(e^{2\varpi_1}Y_{1,1}^2Y_{0,2}^{-2})$ is isomorphic to the connected component of $\mathcal{B}(2\varpi_1)$ generated by $v_{2\varpi_1}$ [HN06, Proposition 3.1]. One checks that the map

$$\mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}) \longrightarrow \mathcal{M}(e^{2\varpi_1}Y_{1,1}^2Y_{0,2}^{-2})$$

which sends the monomial M_s to the extremal element $e^{2\varpi_1}Y_{1,1}^2Y_{0,2}^{-2}$ is an isomorphism of P_{cl}-crystals for all $s \in \mathbb{N}$. Then the result is a direct consequence of the description of the crystal $\mathcal{B}(2\varpi_1)$ given in [BN04]: all the connected components of $\mathcal{B}(2\varpi_1)$ are isomorphic to each other modulo shift of weight by δ. □

2.4.2 Construction of the $\mathcal{U}_q(sl_4^{tor})$-module $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.

Let us give the main result of this section.

Theorem 2.4.6. *There exists a thin representation of $\mathcal{U}_q(sl_4^{tor})$ whose q–character is the sum of monomials occurring in $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ with multiplicity one. It is denoted by $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.*

The construction of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is analogous to the one of $V(e^{\varpi_\ell}Y_{\ell,0}Y_{0,\ell}^{-1})$ in Theorem 2.3.1: we paste together the finite-dimensional representations $V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k})^{(j)}$ and $V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}$ of $\mathcal{U}_q^{v,j}(sl_4^{tor})$ with $j \in I$, $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$.

Let us begin by recalling some well-known facts about the Kirillov-Reshetikhin module $V_0(\Xi^0(M))$ over $\mathcal{U}_q(\hat{sl}_4)'$ with $M = e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}$. It is irreducible as a $\mathcal{U}_q(sl_4)$-module, isomorphic to $V_0(2\Lambda_1)$. In particular, $V_0(\Xi^0(M))$ is an extremal weight module of extremal weight $2\Lambda_1$ and there exist vectors $v_{\phi^j(M)}$ $(j = 0, \dots, 3)$ such that v_M is an ℓ-highest weight vector of $V_0(\Xi^0(M))$ and

$$(x_{i,0}^-)^{(2)} \cdot v_{\phi^{i-1}(M)} = v_{\phi^i(M)} \text{ for } i = 1, \dots, 3,$$
$$(x_{i,0}^+)^{(2)} \cdot v_{\phi^i(M)} = v_{\phi^{i-1}(M)} \text{ for } i = 1, \dots, 3,$$
$$x_{i,0}^\pm \cdot v_{\phi^j(M)} = 0 \text{ in the other cases.}$$

Set

$$v_{\tilde{f}_1\cdot M} := x_{1,0}^- \cdot v_M,\ v_{\tilde{f}_2\tilde{f}_1\cdot M} := x_{2,0}^- x_{1,0}^- \cdot v_M,\ v_{\tilde{f}_3\tilde{f}_2\tilde{f}_1\cdot M} := x_{3,0}^- x_{2,0}^- x_{1,0}^- \cdot v_M,$$

$$v_{\tilde{f}_2\cdot\phi(M)} := x_{2,0}^- \cdot v_{\phi(M)},\ v_{\tilde{f}_3\tilde{f}_2\cdot\phi(M)} := x_{3,0}^- x_{2,0}^- \cdot v_{\phi(M)},$$

$$v_{\tilde{f}_3\cdot\phi^2(M)} := x_{3,0}^- \cdot v_{\phi^2(M)}.$$

These vectors form a basis (v_m) of $V_0(\Xi^0(M))$, indexed by the monomials occurring in $\mathcal{M}_{I_0}(M)$. Furthermore for all $m \in \mathcal{M}_{I_0}(M)$, v_m is an ℓ-weight vector of ℓ-weight $\Xi^0(m)$.

The other finite-dimensional representations of $\mathcal{U}_q(\hat{sl}_4)'$ we have to consider are $V_0(\Xi^0(M_s))$ with $M_s = e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}$ and $s \in \mathbb{N}^*$. The following two points are well-known:

(i) $V_0(\Xi^0(M_s))$ is an irreducible $\mathcal{U}_q(\hat{sl}_4)'$-module isomorphic to $V_0(Y_{1,1}) \otimes V_0(Y_{1,-1-4s})$,

(ii) $\mathrm{Res}(V_0(\Xi^0(M_s)))$ is a completely reducible $\mathcal{U}_q(sl_4)$-module isomorphic to $V_0(2\Lambda_1) \oplus V_0(\Lambda_2)$.

Furthermore there exist vectors $v_{\phi^j(M_s)}$ $(j = 0, \dots, 3)$ such that v_{M_s} is an ℓ-highest weight vector of $V_0(\Xi^0(M_s))$ and

$$(x_{i,0}^-)^{(2)} \cdot v_{\phi^{i-1}(M_s)} = v_{\phi^i(M_s)} \text{ for } i = 1, \dots, 3,$$
$$(x_{i,0}^+)^{(2)} \cdot v_{\phi^i(M_s)} = v_{\phi^{i-1}(M_s)} \text{ for } i = 1, \dots, 3,$$
$$x_{i,0}^\pm \cdot v_{\phi^j(M_s)} = 0 \text{ in the other cases.}$$

To complete this family of vectors to a basis of $V_0(\Xi^0(M_s))$, the following example is used.

Example 2.4.7. *Let $a, b \in \mathbb{Z}$ be such that $a \neq b$ and $a \neq b \pm 2$. Consider the $\mathcal{U}_q(\hat{sl}_2)'$-module $V_0(Y_{1,a}Y_{1,b})$. This module was already studied in [Her10]. We have*

$$\chi_q(V_0(Y_{1,a}Y_{1,b})) = Y_{1,a}Y_{1,b} + Y_{1,a}Y_{1,b+2}^{-1} + Y_{1,a+2}^{-1}Y_{1,b} + Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}.$$

In particular, it was shown that there exists a basis

$$\{v_{Y_{1,a}Y_{1,b}}, v_{Y_{1,a+2}^{-1}Y_{1,b}}, v_{Y_{1,a}Y_{1,b+2}^{-1}}, v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}}\}$$

where the action of the Drinfeld generators on it is given by

$$\begin{array}{rcl}
x_r^+ \cdot v_{Y_{1,a}Y_{1,b}} & = & 0, \\
x_r^- \cdot v_{Y_{1,a}Y_{1,b}} & = & \dfrac{q^{b-1} - q^{a+1}}{q^b - q^a} q^{r(a+1)} v_{Y_{1,a+2}^{-1}Y_{1,b}} + \dfrac{q^{b+1} - q^{a-1}}{q^b - q^a} q^{r(b+1)} v_{Y_{1,a}Y_{1,b+2}^{-1}}, \\
x_r^+ \cdot v_{Y_{1,a+2}^{-1}Y_{1,b}} & = & q^{r(a+1)} v_{Y_{1,a}Y_{1,b}}, \\
x_r^- \cdot v_{Y_{1,a+2}^{-1}Y_{1,b}} & = & q^{r(b+1)} v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}}, \\
x_r^+ \cdot v_{Y_{1,a}Y_{1,b+2}^{-1}} & = & q^{r(b+1)} v_{Y_{1,a}Y_{1,b}}, \\
x_r^- \cdot v_{Y_{1,a}Y_{1,b+2}^{-1}} & = & q^{r(a+1)} v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}}, \\
x_r^+ \cdot v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}} & = & \dfrac{q^{b-1} - q^{a+1}}{q^b - q^a} q^{r(b+1)} v_{Y_{1,a+2}^{-1}Y_{1,b}} + \dfrac{q^{b+1} - q^{a-1}}{q^b - q^a} q^{r(a+1)} v_{Y_{1,a}Y_{1,b+2}^{-1}}, \\
x_r^- \cdot v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}} & = & 0,
\end{array}$$

and with v_m of ℓ-weight m for $m = Y_{1,a}Y_{1,b}, \dots, Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}$. Note that the basis used in [Her10] is renormalized here and we have

$$(x_0^-)^{(2)} \cdot v_{Y_{1,a}Y_{1,b}} = v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}}, \ (x_0^+)^{(2)} \cdot v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}} = v_{Y_{1,a}Y_{1,b}}.$$

As $a \neq b \pm 2$, it is well-known that the $\mathcal{U}_q(\hat{sl}_2)'$-module $V_0(Y_{1,a}Y_{1,b})$ is isomorphic to $V_0(Y_{1,a}) \otimes V_0(Y_{1,b})$. Furthermore the $\mathcal{U}_q(sl_2)$-module $\mathrm{Res}(V_0(Y_{1,a}Y_{1,b}))$ *is not irreducible, but it is cyclic generated by one of vectors $v_{Y_{1,a}Y_{1,b+2}^{-1}}$ or $v_{Y_{1,a+2}^{-1}Y_{1,b}}$.*

Set $M_s^1 = Y_{1,3}^{-1}Y_{1,-1-4s}Y_{2,2}Y_{0,-4s}^{-1}$ and $M_s^2 = Y_{1,1}Y_{1,1-4s}^{-1}Y_{2,-4s}Y_{0,2}^{-1}$. Let $v_{M_s^1}$ and $v_{M_s^2} \in V_0(\Xi^0(M))$ of ℓ-weight M_s^1 and M_s^2 respectively, be such that

$$x_{1,0}^- \cdot v_{M_s} = \frac{q^{-2-4s} - q^4}{q^{-1-4s} - q^3} v_{M_s^1} + \frac{q^{-4s} - q^2}{q^{-1-4s} - q^3} v_{M_s^2}.$$

Set

$$v_{\tilde{f}_2 \cdot M_s^u} := x_{2,0}^- \cdot v_{M_s^u}, \ v_{\tilde{f}_3\tilde{f}_2 \cdot M_s^u} := x_{3,0}^- x_{2,0}^- \cdot v_{M_s^u}$$

with $u = 1, 2$. In the same way, one can define $v_{\phi(M_s^u)}, v_{\tilde{f}_3 \cdot \phi(M_s^u)}$ and $v_{\phi^2(M_s^u)}$ for $u = 1, 2$. We check that these vectors form a basis (v_m) of $V_0(\Xi^0(M_s))$, indexed by the monomials occurring in $\mathcal{M}_{0,0,s}$. Moreover v_m is an ℓ-weight vector of ℓ-weight $\Xi^0(m)$ for all m.

By twisting the action of $\mathcal{U}_q(\hat{sl}_4)'$ on $V_0(Y_{1,1}Y_{1,-1})$ and $V_0(Y_{1,1}Y_{1,-1-4s})$ by $\theta^{(j)}$ and t_b for some $b \in \mathbb{C}^*$, we obtain for all $j \in I, k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$

- the $\mathcal{U}_q^{v,j}(sl_4^{tor})$-modules $V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k})^{(j)}$ which we call modules of type KR below,

– the $\mathcal{U}_q^{v,j}(sl_4^{tor})$-modules $V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}$ which we call modules of type s-TP below. The modules of type s-TP for various $s \in \mathbb{N}^*$ are called modules of type TP.

From the construction done above, we get bases (v_m) of these modules indexed by the monomial crystals $\mathcal{M}_{I_j}(\phi^{j+4k}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}))$ (resp. $\mathcal{M}_{j,k,s}$) with analogous properties as the previous ones. In particular, the action on a vector v_m is completely determined by the action of the horizontal quantum affine subalgebra on it and by its ℓ-weight m.

Let us begin the construction of the $\mathcal{U}_q(sl_4^{tor})$-module $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Denote by $\mathcal{E}$ the set of monomials occurring in $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and for all $j \in I$, $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$, $\mathcal{E}_{j,k,0}$ (resp. $\mathcal{E}_{j,k,s}$) the set of monomials corresponding to $\mathcal{M}^1_{j,k,0}$ (resp. $\mathcal{M}_{j,k,s}$). We have for all $0 \leq j \leq 3$,

$$\mathcal{E} = \bigsqcup_{k\in\mathbb{Z}} \mathcal{E}_{j,k,0} \sqcup \bigsqcup_{k\in\mathbb{Z}, s\in\mathbb{N}^*} \mathcal{E}_{j,k,s}.$$

Let

$$V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{m\in\mathcal{E}} \mathbb{C}v_m$$

be the vector space freely generated by $\mathcal{E}$. For $0 \leq j \leq 3$, $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$, set $V_k^{(j)} = \bigoplus_{m\in\mathcal{E}_{j,k,0}} \mathbb{C}v_m$ (resp. $V_{k,s}^{(j)} = \bigoplus_{m\in\mathcal{E}_{j,k,s}} \mathbb{C}v_m$). Then for all $0 \leq j \leq 3$,

$$V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{k\in\mathbb{Z}} V_k^{(j)} \oplus \bigoplus_{k\in\mathbb{Z}, s\in\mathbb{N}^*} V_{k,s}^{(j)}.$$

For all $j \in I$, we endow $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ with a structure of $\mathcal{U}_q^{v,j}(sl_4^{tor})$-module as follows: for $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$, the vector space $V_k^{(j)}$ (resp. $V_{k,s}^{(j)}$) is isomorphic to $V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k})^{(j)}$ (resp. $V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}$) by identifying the corresponding bases. So $V_k^{(j)}$ (resp. $V_{k,s}^{(j)}$) is endowed with a structure of $\mathcal{U}_q^{v,j}(sl_4^{tor})$-module, and $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ also by direct sum. We denote it by $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)}$.

Proposition 2.4.8. *There exists a $\mathcal{U}_q(sl_4^{tor})$-module structure on $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ such that for all $j \in I$ the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module is isomorphic to $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)}$. Furthermore the q–character of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is*

$$\chi_q(V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})) = \sum_{m\in\mathcal{E}} m,$$

where $\mathcal{E}$ is the set of monomials occurring in $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.

Proof. The process is the same as in Theorem 2.3.1: to define an action of $\mathcal{U}_q(sl_4^{tor})$, we determine the action of the subalgebras $\hat{\mathcal{U}}_i$ for all $i \in I$. For that, let $j \in I$ be such that $j \neq i$. Then the action of $\hat{\mathcal{U}}_i$ on $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is the restriction of the action of $\mathcal{U}_q^{v,j}(sl_4^{tor})$ on $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)}$. We check that this is independent of the choice of $j \neq i$.

Let us show that this action endows $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ with a structure of $\mathcal{U}_q(sl_4^{tor})$-module. For that, we have to distinguish two types of monomials:

– the m such that there is no $s, s' \in \mathbb{N}$ with $s \neq s'$ and $m \in \mathcal{E}_{j,k,s} \cap \mathcal{E}_{j',k',s'}$ for some $0 \leq j, j' \leq 3$ and $k, k' \in \mathbb{Z}$. For such a monomial, the defined action on v_m comes from the same type of modules, i.e. only of modules of type KR or only on modules of type s-TP for one $s \in \mathbb{N}^*$,

– the m such that there is $s, s' \in \mathbb{N}$ with $s \neq s'$ and $m \in \mathcal{E}_{j,k,s} \cap \mathcal{E}_{j',k',s'}$ for some $0 \leq j, j' \leq 3$ and $k, k' \in \mathbb{Z}$. For such a monomial, the defined action on v_m comes from two different types of modules, i.e. of modules of type KR and of type TP or of modules of type s-TP and of type s'-TP with $s \neq s'$.

For the first ones, the same process as in Theorem 2.3.1 (using promotion operator) implies that the defining relations of $\mathcal{U}_q(sl_4^{tor})$ hold on it. For the other ones, this is more complicated. Such a monomial is of the form $m = \phi^{j+4k}(Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1})$ with $0 \leq j \leq 3$, $k \in \mathbb{Z}$, $s \in \mathbb{N}^*$. Promotion operator implies some relations on v_m but not all and we check directly that they are satisfied. We do not detail the calculations here. □

2.4.3 Study of the $\mathcal{U}_q(sl_4^{tor})$-module $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$

Proposition 2.4.9. *The $\mathcal{U}_q(sl_4^{tor})$-module $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is integrable. Moreover, it satisfies property (iv) of Remark 1.3.3.*

Proof. For all $j \in I$, $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is completely reducible as a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$-module and we have

$$V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)} = \bigoplus_{s\in\mathbb{N},k\in\mathbb{Z}} V_0(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}. \tag{2.9}$$

The representations occurring in the direct sum at the right hand side are integrable. Hence $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module. Furthermore the modules of type KR (resp. of type TP) are all isomorphic as $\mathcal{U}_q(sl_4)$-modules and satisfy property (iv) of Remark 1.3.3. Then, it holds for $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, i.e.

$$V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})_{\nu+N\alpha_i} = \{0\} \text{ for all } \nu \in P, i \in I, N >> 0.$$

□

Remark 2.4.10. *The weight spaces of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ are infinite-dimensional and property (iii) of Remark 1.3.3 does not hold. However its ℓ-weight spaces are all of dimension one.*

The main result of this section is the following.

Theorem 2.4.11. *Set $M = e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}$. The representation $V(M)$ is an extremal loop weight module generated by the vector v_M of ℓ-weight M.*

Proof. The third point of Definition 1.7.1 is a consequence of (2.9). For the first two points, we use the following results. □

Lemma 2.4.12. *Let V be a $\mathcal{U}_q(\hat{sl}_{n+1})$-module with basis $(v_m)_{m\in\mathcal{M}'}$ indexed by a subcrystal $\mathcal{M}'$ of $\mathcal{M}$. Assume that $M \in \mathcal{M}'$ is extremal of weight $\mathrm{wt}(M)$ and for all $i \in I$ and $m \in W \cdot M$,*

$$\mathrm{wt}(v_m) = \mathrm{wt}(m),\ x_i^{\pm} \cdot v_m = 0 \text{ and } (x_i^{\mp})^{(\pm\mathrm{wt}(m)(h_i))} \cdot v_m = v_{S_i(m)} \text{ if } \pm\mathrm{wt}(m)(h_i) \geq 0.$$

Then v_M is an extremal vector of weight $\mathrm{wt}(M)$.

Proof. The proof is analogue to the one of Lemma 2.3.8. □

Corollary 2.4.13. *Set $M_s = e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}$ $(s \in \mathbb{N})$. Then v_{M_s} is an extremal vector of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ of weight $2\varpi_1 + s\delta$ for the horizontal quantum affine subalgebra $\mathcal{U}_q^h(sl_4^{tor})$.*

Proof. By construction of the basis (v_m) of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, we have $\mathrm{wt}(v_m) = \mathrm{wt}(m)$ for all m. Furthermore a monomial in $W \cdot M_s$ is of the form $\phi^{j+4k}(M_s)$ with $j \in I_0$ and $k \in \mathbb{Z}$ and we have $\mathrm{wt}(\phi^{j+4k}(M_s)) = 2\Lambda_{j+1} - 2\Lambda_j - 2k\delta$,

$$\begin{aligned}
(x_{j,0}^{+})^{(2)} \cdot v_{\phi^{j+4k}(M_s)} &= v_{\phi^{j-1+4k}(M_s)} = v_{S_j(\phi^{j+4k}(M_s))}, \\
(x_{j+1,0}^{-})^{(2)} \cdot v_{\phi^{j+4k}(M_s)} &= v_{\phi^{j+1+4k}(M_s)} = v_{S_{j+1}(\phi^{j+4k}(M_s))}, \\
x_i^{\pm} \cdot v_{\phi^{j+4k}(M_s)} &= 0 \text{ in the other cases.}
\end{aligned}$$

Hence the hypotheses of the above lemma are satisfied and v_{M_s} is extremal of weight $2\varpi_1 + s\delta$ for the horizontal quantum affine subalgebra $\mathcal{U}_q^h(sl_4^{tor})$. □

Proposition 2.4.14. *The representation $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is cyclic as a $\mathcal{U}_q^h(sl_4^{tor})$-module, generated by the vector $v = v_{e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}}$.*

Proof. Consider the sub-$\mathcal{U}_q^h(sl_4^{tor})$-module V generated by v. By construction of the basis (v_m) of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, $v_m \in V$ for all $m \in \mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. We proceed by a recursive argument: assume that for one $s \in \mathbb{N}$, we have $v_m \in V$ for all $m \in \mathcal{M}(e^{2\varpi_1 - t\delta}Y_{1,1}Y_{1,-1-4t}Y_{0,2}^{-1}Y_{0,-4t}^{-1})$ with $0 \leq t \leq s$. In particular $v_{Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1}}$ is in V and by Example 2.4.7,

$$x_{0,0}^{-} \cdot v_{Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1}} = v_{e^{2\varpi_1-(s+1)\delta}Y_{1,5}Y_{1,-1-4s}Y_{0,6}^{-1}Y_{0,-4s}^{-1}} \in V.$$

In the same way $v_{\phi^k(Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1})}$ and $v_{\phi^k(e^{2\varpi_1-(s+1)\delta}Y_{1,5}Y_{1,-1-4s}Y_{0,6}^{-1}Y_{0,-4s}^{-1})}$ are in V for any $k \in \mathbb{Z}$. But all the v_m with $m \in \mathcal{M}(e^{2\varpi_1-(s+1)\delta}Y_{1,5}Y_{1,-1-4s}Y_{0,6}^{-1}Y_{0,-4s}^{-1})$ can be obtained from these vectors by action of $\mathcal{U}_q^h(sl_4^{tor})$: this is straightforward from Example 2.4.7 and the construction of the basis (v_m). □

Proposition 2.4.15. *The $\mathcal{U}_q(sl_4^{tor})$-module $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is irreducible.*

Proof. Let V be a non trivial sub-$\mathcal{U}_q(sl_4^{tor})$-module of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. As the ℓ-weight spaces are of dimension one, there exists $s \in \mathbb{N}$ and a monomial $m \in \mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ such that $v_m \in V$.

If $s = 0$, we have already shown in the above proof that $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is cyclic generated by v_m and $V = V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Assume that $s \in \mathbb{N}^*$. By Example 2.4.7 and the construction of (v_m), there exists $x \in \mathcal{U}_q^h(sl_4^{tor})$ such that

$$x \cdot v_m = v_{e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}}.$$

Furthermore $\hat{\mathcal{U}}_1 \cdot v_{e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}}$ is the simple ℓ-highest weight $\hat{\mathcal{U}}_1$-module of ℓ-highest weight $Y_{1,1}Y_{1,-1-4s}$ and there exists $y \in \mathcal{U}_q(sl_4^{tor})$ such that

$$y \cdot v_m = v_{Y_{1,1}Y_{1,1-4s}^{-1}Y_{2,-4s}Y_{0,2}^{-1}}$$

with $Y_{1,1}Y_{1,1-4s}^{-1}Y_{2,-4s}Y_{0,2}^{-1} \in \mathcal{M}\left(e^{2\varpi_1+(s-1)\delta}Y_{1,1}Y_{1,-1-4(s-1)}Y_{0,2}^{-1}Y_{0,-4(s-1)}^{-1}\right)$. Repeating this argument, one shows that the vector $v_{e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}}$ is in V. By the above proposition we get $V = V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. □

Proposition 2.4.16. *The $\mathcal{U}_q(\hat{sl}_4)$-module* $\mathrm{Res}(V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}))$ *has a crystal basis isomorphic to $\mathcal{B}(2\varpi_1)$.*

Proof. Set $K = \mathbb{C}(q)$ with q an indeterminate and let A be the subring of K consisting of rational functions in K without pole at $q = 0$. We normalize the basis (v_m) of the $\mathbb{C}(q)$-vector space $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ as follows. For all $m \in \overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, let w_m be the vector defined by

– $w_m = \frac{1}{q}v_m$ if there exists $k \in \mathbb{Z}, s \in \mathbb{N}^*$ such that

$$m = \phi^k(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}),$$

– $w_m = v_m$ otherwise.

Set $\mathcal{B} = (w_m)_m$ and $\mathcal{L} = \bigoplus_m Aw_m$. We check directly that $(\mathcal{L}, \mathcal{B})$ is a crystal basis of the $\mathcal{U}_q(\hat{sl}_4)$-module $\mathrm{Res}(V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}))$, isomorphic to $\mathcal{B}(2\varpi_1)$. We do not detail the calculations. □

Remark 2.4.17. *All these results suggest that* $\mathrm{Res}(V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}))$ *is isomorphic to the extremal weight* $\mathcal{U}_q(\hat{sl}_4)$*-module* $V(2\varpi_1)$*. One expects to prove such a result for all the extremal loop weight modules constructed by the conjectural process given above.*

2.4.4 Finite-dimensional representations at roots of unity

Set $L \geq 1$ and let ϵ be a primitive $(4L)$-root of unity.

Denote by $\mathcal{E}_s$ the set of monomials occurring in $\mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ for all $s \in \mathbb{N}$. Consider $\mathcal{E}'$ the subset of $\mathcal{E}$ defined by

$$\mathcal{E}' = \bigsqcup_{0 \leq s \leq L-1} \mathcal{E}_s.$$

Let $\mathcal{E}_\epsilon$ and $\mathcal{E}'_\epsilon$ be the images of the sets $\mathcal{E}$ and $\mathcal{E}'$ respectively by the map $\Gamma_{(4L)}$: $\mathcal{E}'_\epsilon$ is a finite monomial set of cardinality $16L^2$.

Theorem 2.4.18. *Assume that* ϵ *is a primitive* $4L$*-root of unity. There exists an irreducible* $\mathcal{U}_\epsilon(sl_4^{tor})'$*-module* V_ϵ *of dimension* $16L^2$ *such that*

$$\chi_\epsilon(V_\epsilon) = \sum_{m \in \mathcal{E}'_\epsilon} m.$$

Proof. The main difficulty is to specialize q at ϵ in the $\mathcal{U}_q^{v,j}(sl_4^{tor})$-modules of type TP. In fact, these modules can be undefined or reducible after specialization. For better understand these phenomena, let us study the specialized $\mathcal{U}_\epsilon(\hat{sl}_2)'$-module $V_0(Y_{1,a}Y_{1,b})_\epsilon$ with $a, b \in \mathbb{Z}$. This representation is well defined if $a \notin b + 4L\mathbb{Z}$. Assume that in the following and study $V_0(Y_{1,a}Y_{1,b})_\epsilon$. If $a \notin b \pm 2 + 4L\mathbb{Z}$, this representation is irreducible. If $a \in b + 2 + 4L\mathbb{Z}$, it is not irreducible: in fact

$$\mathcal{U}_\epsilon(\hat{sl}_2)' \cdot v_{Y_{1,a}Y_{1,b}} = \mathbb{C}v_{Y_{1,a}Y_{1,b}} \oplus \mathbb{C}v_{Y_{1,a+2}^{-1}Y_{1,b}} \oplus \mathbb{C}v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}}$$

is an irreducible submodule of $V_0(Y_{1,a}Y_{1,b})_\epsilon$.

By our study of the $\mathcal{U}_\epsilon(\hat{sl}_2)'$-module $V_0(Y_{1,a}Y_{1,b})_\epsilon$, one can specialize q at ϵ in the defining relations of the action on the basis (v_m) of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Moreover one checks that

$$\mathcal{U}_\epsilon(sl_4^{tor})' \cdot v_{Y_{1,1}Y_{1,-1-4L}Y_{0,2}^{-1}Y_{0,-4L}^{-1}} = \bigoplus_{m \in \mathcal{E}-\mathcal{E}'} \mathbb{C}v_m$$

is a sub-$\mathcal{U}_\epsilon(sl_4^{tor})'$-module of $V(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})_\epsilon$. By taking the quotient, we obtain a $\mathcal{U}_\epsilon(sl_4^{tor})'$-module

$$V_\epsilon = \bigoplus_{m\in\mathcal{E}'} \mathbb{C}v_m$$

which is irreducible: this is straightforward from the explicit formulas of the action. $\square$

Chapter 3

Extremal loop weight modules and tensor products for quantum toroidal algebras

This chapter corresponds to the paper [Man13a], preprint arXiv:1305.3481.

3.1 Motivations

3.1.1 Reminders about the extremal loop weight modules

We recall the notion of extremal loop weight modules for $\mathcal{U}_q(sl_{n+1}^{tor})$.

Definition. *(Definition 1.7.1) An extremal loop weight module of $\mathcal{U}_q(sl_{n+1}^{tor})$ of ℓ-weight m is an integrable representation V such that there is $v \in V_m$ satisfying*

(i) $\mathcal{U}_q(sl_{n+1}^{tor}) \cdot v = V$,

(ii) v is extremal for $\mathcal{U}_q^h(sl_{n+1}^{tor})$,

(iii) $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot w$ is finite-dimensional for all $w \in V$ and $j \in I$.

The main motivation of this definition is the construction of finite-dimensional representations of the quantum toroidal algebras as in the theory of Kashiwara, but at roots of unity in this case. We have obtained some results in this sense in Chapter 2. Let

$$d_\ell = \min(\ell, n+1-\ell)$$

be the distance between the nodes 0 and ℓ in the Dynkin diagram of type $A_n^{(1)}$.

Theorem. *(Theorem 2.3.1, Theorem 2.3.7) Assume that $n = 2r+1$ is odd and $\ell = 1, r+1$ or n. Then there exists an irreducible, thin extremal loop weight module of $\mathcal{U}_q(sl_{n+1}^{tor})$ of ℓ-weight $e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1}$. It is denoted by $V(e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1})$ and called extremal fundamental loop weight module.*

The representations $V(e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1})$ are neither of ℓ-highest weight nor of ℓ-lowest weight, and they are not in the category $\mathcal{O}$. Actually as a $\mathcal{U}_q(\hat{sl}_{n+1})$-module, $\mathrm{Res}(V(e^{\varpi_\ell} Y_{\ell,1} Y_{0,q^{d_\ell}}^{-1}))$ is isomorphic to the level 0 extremal fundamental weight module $V(\varpi_\ell)$. The action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on these modules is explicitly given by formulas (2.8).

Its construction is based on monomial realizations of level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$.

For $a \in \mathbb{C}^*$, let us denote by $V(e^{\varpi_\ell}Y_{\ell,a}Y^{-1}_{0,aq^{d_\ell}})$ the $\mathcal{U}_q(sl^{tor}_{n+1})$-module obtained from $V(e^{\varpi_\ell}Y_{\ell,1}Y^{-1}_{0,q^{d_\ell}})$ by twisting the action by the automorphism t_a: they are also ℓ-extremal of ℓ-weight $e^{\varpi_\ell}Y_{\ell,a}Y^{-1}_{0,aq^{d_\ell}}$ and satisfy the same properties.

Denote by $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$ the algebra defined as $\mathcal{U}_q(sl^{tor}_{n+1})'$ with ϵ instead of q (without divided power and derivation element). When q is specialized at a root of unity ϵ we obtain finite-dimensional representations of $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$.

Theorem. *(Theorem 2.3.18) Assume always that $n = 2r+1$ $(r \geq 1)$ is odd. Set $\ell = 1, n$ and $p = n+1$ (resp. $\ell = r+1$ and $p = 2$). Denote by ϵ a primitive $[pL]$-root of unity with $L \geq 1$ (resp. $L > 1$). Then there exists an irreducible $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$-module, denoted by $V(e^{\varpi_\ell}Y_{\ell,a}Y^{-1}_{0,aq^{d_\ell}})_\epsilon$, of dimension $L \times \begin{pmatrix} n+1 \\ \ell \end{pmatrix}$.*

Actually when $\ell = 1$ or $\ell = n$, the representations $V(e^{\varpi_\ell}Y_{\ell,a}Y^{-1}_{0,aq})$ $(a \in \mathbb{C}^*)$ and their specializations at roots of unity $V(e^{\varpi_\ell}Y_{\ell,a}Y^{-1}_{0,aq})_\epsilon$ can be defined for all the quantum toroidal algebras $\mathcal{U}_q(sl^{tor}_{n+1})$ and not only when n is odd. Let us improve this point: consider the monomial realization $\mathcal{M}(e^{\varpi_1}Y_{1,1}Y^{-1}_{0,q})$ of the level 0 extremal weight crystal $\mathcal{B}(\varpi_1)$ of $\mathcal{U}_q(\hat{sl}_{n+1})$

$$\cdots \xrightarrow{0} e^{\Lambda_1-\Lambda_0}Y_{1,1}Y^{-1}_{0,q} \xrightarrow{1} e^{\Lambda_2-\Lambda_1}Y_{2,q}Y^{-1}_{1,q^2} \xrightarrow{2} \cdots \xrightarrow{n} e^{\Lambda_0-\Lambda_n}Y_{0,q^n}Y^{-1}_{n,q^{n+1}} \xrightarrow{0} \cdots$$

The vector space $V(e^{\varpi_1}Y_{1,1}Y^{-1}_{0,q})$ freely generated by these monomials can be endowed with a structure of $\mathcal{U}_q(sl^{tor}_{n+1})$-module by formulas (2.8). Let us denote by $V(e^{\varpi_1}Y_{1,a}Y^{-1}_{0,aq})$ the $\mathcal{U}_q(sl^{tor}_{n+1})$-module obtained by twisting the action by t_a $(a \in \mathbb{C}^*)$. We show as in Chapter 2 that it is an extremal loop weight module of ℓ-weight $e^{\varpi_1}Y_{1,a}Y^{-1}_{0,aq}$. The case $\ell = n$ deduces from the previous one as follows. Let ι be the automorphism of the Dynkin diagram of type $A^{(1)}_n$ such that $\iota(k) = -k$ for all $k \in I$ (where I is identified to the set $\mathbb{Z}/(n+1)\mathbb{Z}$). It defines an automorphism $\iota_\mathfrak{h}$ of $\mathfrak{h}$ by sending h_i, d to $h_{\iota(i)}, d$ for all $i \in I$. Furthermore it induces an algebra automorphism of $\mathcal{U}_q(sl^{tor}_{n+1})$ we still denote ι, which sends $x^\pm_{i,r}, h_{i,m}, k_h$ to $x^\pm_{\iota(i),r}, h_{\iota(i),m}, k_{\iota_\mathfrak{h}(h)}$ $(i \in I, r \in \mathbb{Z}, m \in \mathbb{Z} - \{0\}, h \in \mathfrak{h})$. Then $V(e^{\varpi_n}Y_{n,a}Y^{-1}_{0,aq})$ can be obtained from $V(e^{\varpi_1}Y_{1,a}Y^{-1}_{0,aq})$ by twisting the action on it by ι. In particular, $V(e^{\varpi_n}Y_{n,a}Y^{-1}_{0,aq})$ is also an extremal loop weight module of ℓ-weight $e^{\varpi_n}Y_{n,a}Y^{-1}_{0,aq}$.

With the same technical features, by using monomial realizations of the $\mathcal{U}_q(sl^{tor}_4)$-crystal $\mathcal{B}(2\varpi_1)$, we show

Proposition. *(Theorem 2.4.6, Theorem 2.4.11, Theorem 2.4.18)*

(i) There exists an irreducible extremal loop weight $\mathcal{U}_q(sl^{tor}_4)$-module of ℓ-weight $e^{2\varpi_1}Y_{1,q}Y_{1,q^{-1}}Y^{-1}_{0,q^2}Y^{-1}_{0,1}$. We denote it $V(e^{2\varpi_1}Y_{1,q}Y_{1,q^{-1}}Y^{-1}_{0,q^2}Y^{-1}_{0,1})$ in the following.

(ii) Assume that ϵ is a primitive $[4L]$-root of unity $(L \geq 1)$. There exists an irreducible $\mathcal{U}_\epsilon(sl^{tor}_4)'$-module $V(e^{2\varpi_1}Y_{1,q}Y_{1,q^{-1}}Y^{-1}_{0,q^2}Y^{-1}_{0,1})_\epsilon$ of dimension $[4L]^2$.

In the following, we denote by $V(e^{2\varpi_1}Y_{1,a}Y_{1,aq^{-2}}Y^{-1}_{0,aq}Y^{-1}_{0,aq^{-1}})$ the module obtained from $V(e^{2\varpi_1}Y_{1,q}Y_{1,q^{-1}}Y^{-1}_{0,q^2}Y^{-1}_{0,1})$ by twisting the action by the automorphism $t_{aq^{-1}}$ $(a \in \mathbb{C}^*)$: it is also an extremal loop weight module of ℓ-weight $e^{2\varpi_1}Y_{1,a}Y_{1,aq^{-2}}Y^{-1}_{0,aq}Y^{-1}_{0,aq^{-1}}$.

3.1.2 Abstract

In this chapter, we construct new families of extremal loop weight modules for the quantum toroidal algebras of type A as tensor products of simple ℓ-highest weight modules and simple ℓ-lowest weight modules. This construction involves the Drinfeld "coproduct" and related techniques [FJMM13, Her07]. In general, we conjecture that extremal loop weight modules can be obtained by this process. We prove this conjecture in the following case: we get extremal fundamental loop weight modules, also called specialized vector representations, recently constructed by Feigin-Jimbo-Miwa-Mukhin [FJMM13] and the author [Man12c]. Eventually we define new extremal loop weight modules by tensor products of extremal fundamental loop weight modules. By specializing the quantum parameter, we obtain finite-dimensional modules at roots of unity for quantum toroidal algebras.

3.1.3 Motivations

The extremal loop weight $\mathcal{U}_q(\hat{sl}_{n+1})$-module $V(\lambda)$ ($\lambda \in P$) is closely related to the tensor product of a simple highest weight module and a simple lowest weight module (see [Kas94, Section 8.2]). More precisely, the tensor product

$$V(\lambda_+) \otimes V(\lambda_-)$$

of the $\mathcal{U}_q(\hat{sl}_{n+1})$-modules $V(\lambda_+)$ and $V(\lambda_-)$ is considered in [Kas94] for the study of $V(\lambda)$, where

$$\lambda_+ = \sum_{\lambda(h_i)\geq 0} \lambda(h_i)\Lambda_i \text{ and } \lambda_- = \lambda_+ - \lambda \in P_+.$$

The construction of extremal loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-modules done in this chapter is inspired by the works of Kashiwara: we define the fusion product

$$V(e^{s\Lambda_\ell} Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0} Y_{0,aq^{d_\ell}}^{-s})$$

of the simple ℓ-highest weight module $V(e^{s\Lambda_\ell} Y_{\ell,a}^s)$ and the simple ℓ-lowest weight module $V(e^{-s\Lambda_0} Y_{0,aq^{d_\ell}}^{-s})$ ($s \in \mathbb{N}^*$, $1 \leq \ell \leq n$ and $a \in \mathbb{C}^*$). Our construction involves the Drinfeld coproduct and related features, in the spirit of [FJMM13, Her07]. As in Chapter 2, the main motivation is the construction of finite-dimensional representations of the quantum toroidal algebras at roots of unity.

Some of modules we consider in Section 3.3 are already defined in [FJMM13] for the d-deformation $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ of the quantum toroidal algebra. Let us point out that the construction methods and the motivations are very different than ours: in fact vector representations are defined in [FJMM13] as the quantum version of a module over a Lie algebra of difference operators. It is used to construct inter alia a family of Fock modules of $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ (which are ℓ-lowest weights with natural bases labelled by plane partitions) by a semi-infinite wedge process. More precisely this construction consists in defining an action of $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ on the infinite tensor product of vector representations by inductive limit. In our study of extremal loop weight modules in Chapter 2 and Chapter 3, we consider the quantum toroidal algebras with one parameter of deformation (case $d = 1$). The (specialized) vector representations are introduced in very different ways: by using monomial realizations in Chapter 2, or in this chapter by fusion product of simple ℓ-highest weight modules and simple ℓ-lowest weight modules. Furthermore our goal in this part is to give a process to construct extremal loop weight modules for general quantum affinizations. But d-deformations of quantum toroidal algebras not of type A are not known. It is the main reason why we work with $\mathcal{U}_q(sl_{n+1}^{tor})$ instead of its d-deformation. Indeed we show in Chapter 4 and 5 that it is possible to use the features we develop in Section 3.2 for other quantum affinizations, especially for the quantum toroidal algebra of type D_4.

3.1.4 Summary of results

In this chapter we introduce a tensor product process to define new integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-modules: using the Drinfeld coproduct, we construct the fusion product

$$V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$$

of the simple ℓ-highest weight module $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)$ and the simple ℓ-lowest weight module $V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$ ($s \in \mathbb{N}^*$, $1 \leq \ell \leq n$ and $a \in \mathbb{C}^*$).

Theorem. *(Theorem 3.2.1) The coproduct Δ_D is well-defined on*

$$V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$$

and it endows this tensor product with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module.

Let v^+ (resp. v^-) be the ℓ-highest weight vector of $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)$ (resp. the ℓ-lowest weight vector of $V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$). Let us define the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $T(e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s})$ by

$$T(e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s}) = \mathcal{U}_q(sl_{n+1}^{tor}) \cdot v \subseteq V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s}).$$

with $v = v^+ \otimes v^-$. The main motivation of considering such tensor products is the construction of extremal loop weight modules for $\mathcal{U}_q(sl_{n+1}^{tor})$. But hypotheses on $1 \leq \ell \leq n$ must be taken. In fact we have

Proposition. *(Proposition 3.2.11) A necessary condition for $v \in T(e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s})$ to be extremal for $\mathcal{U}_q^h(sl_{n+1}^{tor})$ is that*

$$(n \textit{ is even and } \ell = 1, n) \textit{ or } (n = 2r+1 \textit{ is odd and } \ell = 1, r+1, n). \tag{C}$$

If (C) is satisfied, we conjecture that we get extremal loop weight modules of $\mathcal{U}_q(sl_{n+1}^{tor})$.

Conjecture. *(Conjecture 3.2.13) Fix $a \in \mathbb{C}^*$ and $s \in \mathbb{N}^*$ and assume that ℓ satisfies (C). Then $T(e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s})$ is an extremal loop weight module of $\mathcal{U}_q(sl_{n+1}^{tor})$ generated by the vector v of ℓ-weight $e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s}$. Further if $s = 1$, $T(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq^{d_\ell}}^{-1})$ is isomorphic to the extremal fundamental loop weight module $V(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq^{d_\ell}}^{-1})$.*

We prove this conjecture when $\ell = 1, n$ and $s = 1$[1].

Proposition. *(Proposition 3.2.16) Set $\ell = 1$ or $\ell = n$. The $\mathcal{U}_q(sl_{n+1}^{tor})$-module $T(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq}^{-1})$ is an extremal loop weight module of ℓ-weight $e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq}^{-1}$. It is isomorphic to the extremal fundamental loop weight module $V(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq}^{-1})$.*

1. We also studied the case $\ell = 1$ and $s = 2$ for $\mathcal{U}_q(sl_4^{tor})$. The first computations done suggest that the conjecture holds in this case. However it is quickly impossible to explicit the action of $\mathcal{U}_q(sl_4^{tor})$, because of the growth of multiplicities in the involving ℓ-weights.

The representation $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ is also called the *(specialized) vector representation* in the following.

We consider the tensor product of k extremal fundamental loop weight modules associated to ϖ_ℓ with n, ℓ satisfying (C), and when the spectral parameters are chosen generic. Set $p_1 = p_n = n+1$, and $p_{r+1} = 2$ if $n = 2r+1$.

Theorem. *(Theorem 3.3.2, Theorem 3.4.3) Assume that n and ℓ satisfy (C). Let $k \in \mathbb{N}^*, a_1, a_2, \cdots, a_k \in \mathbb{C}^*$ be such that*

$$\frac{a_i}{a_j} \notin q^{\pm 2 + p_\ell \mathbb{Z}} \text{ for all } i < j.$$

Then the coproduct Δ_D endows the tensor product

$$V_{\ell,a_1} \otimes V_{\ell,a_2} \otimes \cdots \otimes V_{\ell,a_k}$$

with a structure of irreducible representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ which is ℓ-extremal of ℓ-weight

$$Y_{\ell,a_1} \cdots Y_{\ell,a_k} Y_{0,a_1q^{d_\ell}}^{-1} \cdots Y_{0,a_1q^{d_\ell}}^{-1}.$$

We consider the case of tensor products of vector representations when the set of spectral parameters form a q-segment. We get in this way another family of extremal loop weight modules.

Proposition. *(Proposition 3.3.7) Set $k \in \mathbb{N}^*$ and $a \in \mathbb{C}^*$. There exists a subvector space $V_{1,\boldsymbol{a}}$ (where $\boldsymbol{a} = \{a, aq^{-2}, \cdots, aq^{-2(k-1)}\}$) of*

$$V_{1,a} \otimes V_{1,aq^{-2}} \otimes \cdots \otimes V_{1,aq^{-2(k-1)}}$$

which can be endowed with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module. Furthermore $V_{1,\boldsymbol{a}}$ is irreducible and ℓ-extremal of ℓ-weight

$$Y_{1,a}Y_{1,aq^{-2}} \cdots Y_{1,aq^{-2(k-1)}} Y_{0,aq}^{-1} Y_{0,aq^{-1}}^{-1} \cdots Y_{0,aq^{-2(k-1)+1}}^{-1}.$$

In particular we recover the extremal loop weight $\mathcal{U}_q(sl_4^{tor})$-module constructed in Section 2.4.

Proposition. *(Proposition 3.3.8) The $\mathcal{U}_q(sl_4^{tor})$-modules $V(e^{2\varpi_1}Y_{1,a}Y_{1,aq^{-2}}Y_{0,aq}^{-1}Y_{0,aq^{-1}}^{-1})$ and $V_{1,\{a,aq^{-2}\}}$ are isomorphic.*

We obtain also a new construction of the extremal fundamental loop weight modules.

Theorem. *(Theorem 3.3.13) Assume that $n = 2r+1$ is odd ($r \geq 1$). The coproduct Δ_D endows the tensor product*

$$V_{1,a} \otimes V_{1,aq^{-2}} \otimes \cdots \otimes V_{1,aq^{-2r}}$$

with a $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure. Furthermore this representation is not irreducible, and admits a submodule isomorphic to the extremal fundamental loop weight module $V_{r+1,aq^{-r-2}}$.

By specializing the quantum parameter, we get new finite-dimensional representations of the quantum toroidal algebras at roots of unity.

Theorem. *(Theorem 3.3.15, Theorem 3.4.4) Assume that n and ℓ satisfy (C). Set $L \geq 1$ if $\ell = 1, n$ (resp. $L > 1$ if $n = 2r+1$ and $\ell = r+1$). Let ϵ be a primitive $[p_\ell L]$-root of unity and $k \in \mathbb{N}^*, a_1, a_2, \cdots, a_k \in \mathbb{C}^*$ such that*

$$\frac{a_i}{a_j} \notin \epsilon^{\pm 2 + p_\ell \mathbb{Z}} \text{ for all } i < j.$$

Then the coproduct Δ_D endows the tensor product

$$[V_{\ell,a_1}]_\epsilon \otimes [V_{\ell,a_2}]_\epsilon \otimes \cdots \otimes [V_{\ell,a_k}]_\epsilon$$

with a structure of simple $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$-module which is finite-dimensional of dimension $\left[\begin{pmatrix} n+1 \\ \ell \end{pmatrix} \times L\right]^k$.

An analogue result is also obtained for the tensor product of vector representations when the set of spectral parameters forms a q-segment (Theorem 3.3.16).

3.1.5 Organization

This chapter is organized as follows.

In Section 3.2 we introduce a tensor product process to define new integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-modules: we study tensor products of simple ℓ-highest weight modules and simple ℓ-lowest weight modules associated respectively to the weights $s\Lambda_\ell$ and $-s\Lambda_0$ ($s \in \mathbb{N}^*, 1 \leq \ell \leq n$). We show that, for a particular choice of non-zero complex parameters, the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ is well-defined on it (Theorem 3.2.1). We conjecture that we obtain in this way an extremal loop weight module associated to the weight $s\varpi_\ell = s\Lambda_\ell - s\Lambda_0$ if and only if ℓ satisfies (C) (Conjecture 3.2.13). We prove the conjecture for $s = 1$ and $\ell = 1, n$. When $\ell, s = 1$, the module hence obtained is isomorphic to the extremal fundamental loop weight module associated to the weight ϖ_1, also called specialized vector representation (Proposition 3.2.16).

Section 3.3 is devoted to the study of tensor products of specialized vector representations. When the non-zero complex parameters are generic, we check that the $\mathcal{U}_q(sl_{n+1}^{tor})$-module hence obtained is an extremal loop weight module (Theorem 3.3.2). When the non-zero complex parameters form a q-segment, we determine conditions for the action to be well-defined (Proposition 3.3.1). We get extremal loop weight modules associated to a multiple of the weight ϖ_1 (Theorem 3.3.5 and Proposition 3.3.7). Furthermore we recover the extremal fundamental loop weight modules defined in [Man12c] (Theorem 3.3.13).

In Section 3.4 we consider tensor products of extremal fundamental loop weight modules associated to the weight $\varpi_\ell = \Lambda_\ell - \Lambda_0$ when $n = 2r+1$ is odd and $\ell = r+1$. We determine when the action is well-defined on it (Proposition 3.4.1). When the non-zero complex parameters are generic we show that the tensor product is an irreducible extremal loop weight module (Theorem 3.4.3).

3.2 Tensor product of ℓ-highest weight modules and ℓ-lowest weight modules

In this section, we introduce a tensor product process to define new integrable modules over the quantum toroidal algebra. We define a $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure on the tensor product of simple ℓ-highest weight modules and simple ℓ-lowest weight modules associated to the weights $s\Lambda_\ell$ and $-s\Lambda_0$ respectively ($s \in \mathbb{N}^*, 1 \leq \ell \leq n$). The main motivation for considering such tensor products comes from Remark 1.4.6: we expect to obtain other extremal loop weight modules by this process. To do that we use the coproduct Δ_D. But it involves infinite sums and it cannot endow every tensor product of ℓ-highest weight modules and ℓ-lowest weight modules with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module. In fact convergence properties depend on the non-zero complex parameters defining the ℓ-weight modules. In particular, they have to satisfy restrictive conditions. For these reasons we made the choice to consider the following tensor product in this section

$$V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s}) \text{ with } 1 \leq \ell \leq n, a \in \mathbb{C}^* \text{ and } s \geq 1.$$

We will see that the convergence properties are well satisfied in this case, and that it is a good candidate to obtain extremal loop weight modules. Let us note that the general process we introduce in this section can be used to define new integrable modules in many other situations (see also Remark 3.2.7 thereupon).

In the first subsection, we show that the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)\otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$ given by the coproduct Δ_D is well-defined (Theorem 3.2.1).

In the second subsection, we consider the sub-$\mathcal{U}_q(sl_{n+1}^{tor})$-module

$$T(e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s}) = \mathcal{U}_q(sl_{n+1}^{tor}) \cdot \left(v^+ \otimes v^-\right) \subseteq V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$$

where v^+ (resp. v^-) is the ℓ-highest weight vector of $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)$ (resp. the ℓ-lowest weight vector of $V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$). We determine necessary conditions on ℓ for $T(e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s})$ to be an extremal loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module: it requires that

$$(\ell = 1, n) \text{ or } (n = 2r + 1 \text{ and } \ell = r + 1). \tag{C}$$

We conjecture that in these cases $T(e^{s\varpi_\ell}Y_{\ell,a}^sY_{0,aq^{d_\ell}}^{-s})$ is an extremal loop weight module, and $T(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq^{d_\ell}}^{-1})$ is isomorphic to the extremal fundamental loop weight module $V(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq^{d_\ell}}^{-1})$ we defined in Part I (Conjecture 3.2.13).

We prove this conjecture for $\ell = 1, n$ and $s = 1$ in the last subsection (Proposition 3.2.16). For $\ell = 1$, the representations are also defined in [FJMM13] for the algebra $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ and called vector representations. We obtain here a specialization at $d = 1$ of them. For this reason we call $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ the *specialized vector representation.*

3.2.1 Existence of the action on the tensor product

Let $\ell \in I_0$, $a \in \mathbb{C}^*$ and $s \in \mathbb{N}^*$. Consider the tensor product $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)\otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$-modules. Let us begin by the main result of this section.

Theorem 3.2.1. *The coproduct Δ_D is well-defined on $V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$ and it endows this tensor product with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module.*

To prove this theorem, we need some intermediary results. Let us begin by the following proposition.

Proposition 3.2.2. *[Her07] Let V be a finite-dimensional $\mathcal{U}_q(\hat{sl}_2)'$-module and fix $m \in \mathbb{Z}$. There are finite numbers of $\mathbb{C}$-linear operators $A_{k,\lambda_+}, \cdots, F_{k,\lambda_-} : V \to V$ $(k \geq 0,\ \lambda_\pm \in \mathbb{C}^*)$ such that for all $r \geq 0$, $s \geq 1$ and $v \in V$,*

$$x^+_{1,m+r} \cdot v = \sum_{k\geq 0, \lambda_+ \in \mathbb{C}^*} \lambda_+^r r^k A_{k,\lambda_+}(v)\ ,\ x^+_{1,m-s} \cdot v = \sum_{k\geq 0, \lambda_- \in \mathbb{C}^*} \lambda_-^s s^k B_{k,\lambda_-}(v), \tag{3.1}$$

$$x^-_{1,m+r} \cdot v = \sum_{k\geq 0, \lambda_+ \in \mathbb{C}^*} \lambda_+^r r^k C_{k,\lambda_+}(v)\ ,\ x^-_{1,m-s} \cdot v = \sum_{k\geq 0, \lambda_- \in \mathbb{C}^*} \lambda_-^s s^k D_{k,\lambda_-}(v), \tag{3.2}$$

$$\phi^+_{1,s} \cdot v = \sum_{k\geq 0, \lambda_+ \in \mathbb{C}^*} \lambda_+^s s^k E_{k,\lambda_+}(v)\ ,\ \phi^-_{1,-s} \cdot v = \sum_{k\geq 0, \lambda_- \in \mathbb{C}^*} \lambda_-^s s^k F_{k,\lambda_-}(v). \tag{3.3}$$

The proof of Proposition 3.2.2 will be useful in the following. We recall it for the convenience of the reader.

Proof. Let us begin by the following well-known identity ($r \in \mathbb{Z} - \{0\}$, $m \in \mathbb{Z}$)

$$[h_{1,r}, x^\pm_{1,m}] = \pm \frac{[2r]_q}{r} x^\pm_{1,m+r}. \tag{3.4}$$

Denote by $\rho : \mathcal{U}_q(\hat{sl}_2)' \to \mathrm{End}(V)$ the action and set

$$\Phi : \mathrm{End}(V) \to \mathrm{End}(V), \Phi = \frac{1}{[2]_q} \mathrm{ad}(h_{1,1}).$$

Consider the Jordan decomposition $\Phi = \Phi_1 + \Phi_2$ of Φ, where Φ_1 is diagonalizable, $\Phi_2^S = 0$ for some $S \geq 1$ and $\Phi_1 \circ \Phi_2 = \Phi_2 \circ \Phi_1$. By (3.4) we have for $m \geq 0$,

$$\rho(x^+_{1,m+r}) = \Phi^r(\rho(x^+_{1,m})) = \sum_{0\leq s\leq S} \begin{bmatrix} r \\ s \end{bmatrix} (\Phi_2^s \Phi_1^{r-s})(\rho(x^+_{1,m})) = \sum_{\lambda \in \mathbb{C}} \sum_{0\leq s \leq S} \begin{bmatrix} r \\ s \end{bmatrix} \lambda^{r-s} (\Phi_2^s)(d_\lambda)$$

where $\rho(x^+_{1,m}) = \sum_{\lambda\in\mathbb{C}} d_\lambda$ is decomposed as a sum of eigenvectors of Φ_1 such that $\Phi_1(d_\lambda) = \lambda d_\lambda$. As $\begin{bmatrix} r \\ s \end{bmatrix}$ is a polynomial in r, there exist linear operators $A_{k,\lambda_+} : V \to V$ such that

$$x^+_{1,m+r} \cdot v = \sum_{k\geq 0} \sum_{\lambda_+ \in \mathbb{C}} \lambda_+^r r^k A_{k,\lambda_+}(v).$$

For $s \geq 1$ and the operators $x^+_{1,m-s}$, we have to replace Φ by $\Phi' = \frac{1}{[2]_q}\mathrm{ad}(h_{1,-1})$. We obtain the result for the $x^-_{1,m}$ in the same way, and for the $\phi^\pm_{1,\pm s}$ by computing formulas for $x^+_{1,\pm s}$ by $-\mathrm{ad}(x^-_{1,0})$. □

Let us improve Proposition 3.2.2. For that we need the following well-known lemma.

Lemma 3.2.3. *[FR99] Let V be a finite-dimensional $\mathcal{U}_q(\hat{sl}_2)'$-module and $m = Y_{1,a_1} \cdots Y_{1,a_k} Y^{-1}_{1,b_1} \cdots Y^{-1}_{1,b_l}$ an ℓ-weight of V with $a_1, \cdots, a_k, b_1, \cdots, b_l \in \mathbb{C}^*$. Denote by $\lambda_\pm(m)$ the eigenvalue of the operators $h_{1,\pm 1}$ on V_m. Then we have*

$$\lambda_\pm(m) = \sum_{1\leq u\leq k} a_u^{\pm 1} - \sum_{1\leq v\leq l} b_v^{\pm 1}.$$

Proposition 3.2.4. *Let V be a finite-dimensional $\mathcal{U}_q(\hat{sl}_2)'$-module. Then the $\mathbb{C}$-linear operators $A_{k,\lambda_+}, \cdots, F_{k,\lambda_-}$ occurring in formulas (3.1), (3.2) and (3.3) are non-zero only if there exist $m, m' \in \mathcal{M}(V)$ (= the set of ℓ-weights of V) such that*

$$\lambda_\pm = \lambda_\pm(m) - \lambda_\pm(m').$$

Proof. To prove this result, let us begin to note that in the proof of Proposition 3.2.2, the complex numbers $\lambda_\pm \in \mathbb{C}^*$ for which the $\mathbb{C}$-linear operators $A_{k,\lambda_+}, \cdots, F_{k,\lambda_-} : V \to V$ ($k \geq 0$) are non-zero, are eigenvalues of $\Phi = \frac{1}{[2]_q}\mathrm{ad}(h_{1,1})$ and $\Phi' = \frac{1}{[2]_q}\mathrm{ad}(h_{1,-1})$. So we have to study the spectrum of the operators $\mathrm{ad}(h_{1,\pm1})$.

By a standard result of Lie algebras an eigenvalue $\lambda_\pm$ of $\mathrm{ad}(h_{1,\pm1})$ is of the form $\lambda_\pm = \alpha_\pm - \beta_\pm$ with $\alpha_\pm, \beta_\pm$ eigenvalues of $h_{1,\pm1}$. So by Lemma 3.2.3, the complex numbers $\lambda_\pm$ we have to consider are of the form ($m, m' \in \mathcal{M}(V)$)

$$\lambda_\pm = \lambda_\pm(m) - \lambda_\pm(m').$$

□

Fix $1 \leq \ell \leq n$, $a \in \mathbb{C}^*$, $s \in \mathbb{N}^*$ and consider the simple ℓ-highest weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)$. For $0 \leq i \leq n$ and $v \in V(e^{s\Lambda_\ell}Y_{\ell,a}^s)$, we consider the finite-dimensional $\hat{\mathcal{U}}_i$-module $W = \hat{\mathcal{U}}_i \cdot v$.

Let $d : I \times I \to \mathbb{N}$ be the map such that for all $0 \leq i, j \leq n$, $d(i,j)$ is the natural number defined by

$$0 \leq d(i,j) \leq \left[\frac{n+1}{2}\right] \text{ and } i = j \pm d(i,j) \mod (n+1)$$

where $[x]$ is the floor of the real number x.

Proposition 3.2.5. *The $\mathbb{C}$-linear operators A_{k,λ_+}, C_{k,λ_+}, E_{k,λ_+} (resp. B_{k,λ_-}, D_{k,λ_-}, F_{k,λ_-}) occurring in formulas (3.1), (3.2) and (3.3) for the $\hat{\mathcal{U}}_i$-module W are non-zero only if*

$$\lambda_+ \in aq^{d(i,\ell)} \cdot (\mathbb{N} + q\mathbb{Z}[q]),$$
$$\lambda_- \in \frac{1}{aq^{d(i,\ell)}} \cdot (\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]).$$

Proof. By the last proposition, we have to study the ℓ-weights of $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)$. The theory of q–characters for the simple ℓ-highest weight $\mathcal{U}_q(sl_{n+1}^{tor})$-modules (see [FR99, Her05, Her07, Nak01]) shows that the highest integers k, l such that variables Y_{i,aq^k} and Y_{i,aq^l}^{-1} occur in $\chi_q(V(e^{s\Lambda_\ell}Y_{\ell,a}^s))$ are for $k = d(i,\ell)$ and $l = d(i,\ell) + 2$.

It remains to determine the positivity of the factors $a^{\pm1}q^{\pm d(i,\ell)}$ in $\lambda_\pm$. For that fix $r \in \mathbb{Z}$ and let us improve the decomposition $\rho(x_{i,r}^+) = \sum_{\lambda_+ \in \mathbb{C}} d_{\lambda_+}$ as eigenvectors of $\Phi : W \to W$ used in the proof of Proposition 3.2.2 (the proof is the same for λ_- by considering Φ'). Actually a complex number λ_+ occurring in this sum satisfies

$$\lambda_+ = \lambda_+(m') - \lambda_+(m)$$

where $m, m' \in \mathcal{M}(W)$ are such that $(x_{i,r}^+ \cdot W_m) \cap W_{m'} \neq \{0\}$. In fact this eigenvalue corresponds to the commutator $[h_{i,1}, x_{i,r}^+]$. By the theory of q–characters, the power of the indeterminate Y_{i,aq^k} in m' is higher than the one in m, and the other complex numbers $b \in \mathbb{C}^*$ such that $Y_{i,b}^{\pm1}$ occur in m or m' satisfy $b \in aq^{d(i,\ell)+1+\mathbb{N}}$. Hence we have

$$\lambda_+ \in aq^{d(i,\ell)} \cdot (\mathbb{N} + q\mathbb{Z}[q]).$$

We treat the other cases in the same way. □

Remark 3.2.6. *Let us consider the simple ℓ-lowest weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{-s\Lambda_0}Y_{0,aq^\ell}^{-s})$ ($1 \leq \ell \leq n$, $a \in \mathbb{C}^*$, $s \in \mathbb{N}^*$) and set $0 \leq i \leq n$. For a vector $v \in V(e^{-s\Lambda_0}Y_{0,aq^\ell}^{-s})$, we consider the finite-dimensional $\hat{\mathcal{U}}_i$-module $W = \hat{\mathcal{U}}_i \cdot v$. We show in the same way that the $\mathbb{C}$-linear operators $A_{k,\lambda_+}, C_{k,\lambda_+}, E_{k,\lambda_+}$ (resp. $B_{k,\lambda_-}, D_{k,\lambda_-}, F_{k,\lambda_-}$) occurring in formulas (3.1), (3.2) and (3.3) for the $\hat{\mathcal{U}}_i$-module W are non-zero only if*

$$\lambda_+ \in aq^{d(i,\ell)} \cdot (-\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]),$$

$$\lambda_- \in \frac{1}{aq^{d(i,\ell)}} \cdot (-\mathbb{N} + q\mathbb{Z}[q]).$$

Now we are able to prove Theorem 3.2.1.

Proof. We assume in the following that $\ell \leq \left[\frac{n+1}{2}\right]$ (the case $\ell > \left[\frac{n+1}{2}\right]$ is deduced from the previous one by using the automorphism ι of $\mathcal{U}_q(sl_{n+1}^{tor})$). We consider the tensor product

$$V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^\ell}^{-s})$$

with $s \in \mathbb{N}^*$ and $a \in \mathbb{C}^*$. Fix $0 \leq i \leq n$, and set $u = v \otimes w$ with $v \in V(e^{s\Lambda_\ell}Y_{\ell,a}^s)$ and $w \in V(e^{-s\Lambda_0}Y_{0,aq^\ell}^{-s})$ ($1 \leq \ell \leq n$, $a \in \mathbb{C}^*$, $s \in \mathbb{N}^*$). Consider the action of $x_i^+(z)$ on the tensor product $V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^\ell}^{-s})$

$$x_i^+(z) \cdot u \quad = \quad (x_i^+(z) \cdot v) \otimes w + (\phi_i^-(z) \cdot v) \otimes (x_i^+(z) \cdot w).$$

We have to prove that at the right hand side, the last term which involves infinite sums is in fact well-defined. Let us consider the term of degree $m \in \mathbb{Z}$ in this sum. It is equal to

$$\begin{aligned}\sum_{r \geq 0} (\phi_{i,-r}^- \cdot v) \otimes (x_{i,m+r}^+ \cdot w) \quad &= \quad (k_i^{-1} \cdot v) \otimes (x_{i,m}^+ \cdot w) \\ &\quad + \sum_{r \geq 1} (\phi_{i,-r}^- \cdot v) \otimes (x_{i,m+r}^+ \cdot w).\end{aligned}$$

As the two considered modules are integrable, we can apply the preceding results. With notations used above, the infinite sum at the right hand side is equal to

$$\sum_{k \geq 0, \lambda_- \in \mathbb{C}^*} \sum_{l \geq 0, \lambda_+ \in \mathbb{C}^*} \left(\sum_{r \geq 1} \lambda_-^r \lambda_+^r r^{k+l} \right) F_{k,\lambda_-}(v) \otimes A_{l,\lambda_+}(w). \tag{3.5}$$

The infinite sum $\sum_{r \geq 1} \lambda_-^r \lambda_+^r r^{k+l}$ is a rational fraction on the variables $\lambda_\pm$ if and only if

$$\lambda_- \cdot \lambda_+ \neq 1$$

for all $\lambda_\pm$ occurring in (3.5). It holds for the tensor product $V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^\ell}^{-s})$: in fact by Proposition 3.2.5,

$$\lambda_- \in \frac{1}{aq^{d(i,\ell)}} \cdot (\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) \text{ and } \lambda_+ \in aq^{d(i,\ell)} \cdot (-\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]).$$

As q is generic the produce $\lambda_- \cdot \lambda_+$ is different to one. Hence the coefficients of the series $x_i^+(z) \cdot u$ are well-defined. We proceed in the same way for the operators $x_i^-(z)$, $0 \leq i \leq n$. So by Lemma 1.8.1, $V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^\ell}^{-s})$ has a $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure. $\square$

Remark 3.2.7. *To define the $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure on $V(e^{s\Lambda_\ell}Y_{\ell,a}^s)\otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$ ($1 \leq \ell \leq n, a \in \mathbb{C}^*, s \in \mathbb{N}^*$), the main step has been to study the ℓ-weights of each involving module (or equivalently their q–character - see the proof of Proposition 3.2.5). Many other $\mathcal{U}_q(sl_{n+1}^{tor})$-modules can be constructed by the same technical feature. Let us give two other examples.*

– *Let us consider the tensor product of $\mathcal{U}_q(sl_{n+1}^{tor})$-modules ($1 \leq \ell \leq n, a \in \mathbb{C}^*$, $s \in \mathbb{N}^*$)*

$$V(e^{s\Lambda_\ell}Y_{\ell,aq^u}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell - v}}^{-s}) \textit{ with } u, v \in \mathbb{N}.$$

Then one can show (the proof is similar to the one of Theorem 3.2.1) that Δ_D defines a $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure on it.

– *Another situation which could be interesting to study is the tensor product of ℓ-highest weight Kirillov-Reshetikhin modules and ℓ-lowest weight Kirillov-Reshetikhin modules. For example, let us consider the tensor product of $\mathcal{U}_q(sl_{n+1}^{tor})$-modules*

$$V = V(e^{2\Lambda_1}Y_{1,q}Y_{1,q^{-1}}) \otimes V(e^{-2\Lambda_0}Y_{0,q^2}^{-1}Y_{0,1}^{-1}).$$

The q–character of these modules is known in terms of tableaux [Her11]. So one can apply the technical feature we used above for this tensor product. But it is quite more difficult than the one studied above. In fact, our study implies that the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ is well-defined on almost all V. And for a finite number of vectors, a direct computation has to be done. We do not detail the calculations here.

Remark 3.2.8. *Our construction holds for all the quantum affinizations, in particular for quantum toroidal algebras of other types. We expect to construct in this way new families of extremal loop weight modules. We give an example of such construction in Chapter 5 for the quantum toroidal algebra of type D_4 (Theorem 5.1.2).*

Remark 3.2.9. *We have explained above our motivations to work with $\mathcal{U}_q(sl_{n+1}^{tor})$ instead of its d-deformation $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ (which is considered in [FJMM13]). However it gives some additional difficulties in our computations. In fact for $\mathcal{U}_q(sl_{n+1}^{tor})$, the fundamental modules are not weighted in general (see [Her09, Section 4.1]) and we have to deal with generalized eigenspaces. The representation theory of $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ is very different: all the simple ℓ-highest weight modules are weighted. This is also the case for the representations considered in [FJMM13] which are all thin. For tensor products of such representations, necessary and sufficient conditions for Δ_D to define an action of $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ can be determined. For these reasons, one can expect to improve Theorem 3.2.1 in the $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ situation. We plan to discuss this point in a subsequent publication.*

3.2.2 Study of the tensor product

Consider the tensor product of $\mathcal{U}_q(sl_{n+1}^{tor})$-modules

$$V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s}) \text{ with } 1 \leq \ell \leq n, a \in \mathbb{C}^* \text{ and } s \in \mathbb{N}^*.$$

By using the coproduct Δ_D, it can be endowed with a $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure (Theorem 3.2.1).

Proposition 3.2.10. *The $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V = V(e^{s\Lambda_\ell}Y_{\ell,a}^s) \otimes V(e^{-s\Lambda_0}Y_{0,aq^{d_\ell}}^{-s})$ is integrable and satisfies*

$$\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot u \textit{ is finite-dimensional for all } u \in V \textit{ and } j \in I.$$

Proof. By Lemma 1.8.3, V is ℓ-integrable. In particular it is integrable. Furthermore $V(e^{s\Lambda_\ell}Y^s_{\ell,a})$ and $V(e^{-s\Lambda_0}Y^{-s}_{0,aq^{d_\ell}})$ are extremal loop weight modules by Proposition 1.7.2. Then by Remark 1.8.4, $\mathcal{U}^{v,j}_q(sl^{tor}_{n+1}) \cdot u$ is finite-dimensional for all $u \in V$ and $j \in I$. $\square$

Set v^+ (resp. v^-) the ℓ-highest weight vector of $V(e^{s\Lambda_\ell}Y^s_{\ell,a})$ (resp. the ℓ-lowest weight vector of $V(e^{-s\Lambda_0}Y^{-s}_{0,aq^{d_\ell}})$). Then $v = v^+ \otimes v^-$ is of ℓ-weight $e^{s\varpi_\ell}Y^s_{\ell,a}Y^{-s}_{0,aq^{d_\ell}}$. Let us define the $\mathcal{U}_q(sl^{tor}_{n+1})$-module $T(e^{s\varpi_\ell}Y^s_{\ell,a}Y^{-s}_{0,aq^{d_\ell}})$ by

$$T(e^{s\varpi_\ell}Y^s_{\ell,a}Y^{-s}_{0,aq^{d_\ell}}) = \mathcal{U}_q(sl^{tor}_{n+1}) \cdot v \subseteq V(e^{s\Lambda_\ell}Y^s_{\ell,a}) \otimes V(e^{-s\Lambda_0}Y^{-s}_{0,aq^{d_\ell}}).$$

The main motivation of considering such tensor products is the construction of extremal loop weight modules for $\mathcal{U}_q(sl^{tor}_{n+1})$. By the last proposition, $T(e^{s\varpi_\ell}Y^s_{\ell,a}Y^{-s}_{0,aq^{d_\ell}})$ is an integrable $\mathcal{U}_q(sl^{tor}_{n+1})$-module which satisfies the first point and the third point of Definition 1.7.1. So it remains to show that v is extremal for the horizontal quantum affine subalgebra. For that hypotheses on $1 \leq \ell \leq n$ must be taken. In fact we have

Proposition 3.2.11. *Assume that*

$$(n \text{ is even and } \ell \neq 1, n) \text{ or } (n = 2r+1 \text{ is odd and } \ell \neq 1, r+1, n).$$

Then $v \in T(e^{s\varpi_\ell}Y^s_{\ell,a}Y^{-s}_{0,aq^{d_\ell}})$ is not extremal for the horizontal quantum affine subalgebra.

To prove this result, the following identity will be useful

$$\gamma(z)\delta(z/w) = \gamma(w)\delta(z/w)$$

where $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$ and $\gamma(z)$ is any rational function (it occurs also in [FJMM13]). We will often use it in the whole thesis.

Proof. One can assume that $\ell \leq \left[\frac{n+1}{2}\right]$ (we use the automorphism ι for the case $\ell > \left[\frac{n+1}{2}\right]$). Let us show that v is not extremal for the horizontal quantum affine subalgebra under these hypotheses on ℓ. By a direct computation, we check that for all $i \in I$ and $\ell \leq t \leq n-1$, $S_t \circ \cdots \circ S_\ell(v)$ is i-extremal and

$$S_n \circ \cdots \circ S_\ell(v) \in \mathbb{C}^* \cdot \Big(S_n \circ \cdots \circ S_\ell(v^+)\Big) \otimes v^-.$$

The vector $w = S_n \circ \cdots \circ S_\ell(v^+)$ is of ℓ-weight $Y^s_{\ell-1,aq}Y^{-s}_{n,aq^{n+2-\ell}}Y^s_{0,aq^{n+1-\ell}}$ and $w \otimes v^-$ is of ℓ-weight $Y^s_{\ell-1,aq}Y^{-s}_{n,aq^{n+2-\ell}}Y^s_{0,aq^{n+1-\ell}}Y^{-s}_{0,aq^\ell}$. By hypothesis we have $q^{n+3-2\ell} \neq q^2$ and $q^{2\ell-1-n} \neq 1$. So we obtain

$$\begin{aligned}
x^+_{0,0} \cdot w \otimes v^- &= \psi(q^{n+3-2\ell})^s \delta(aq^{\ell-1}z) \cdot w \otimes (x^+_{0,0} \cdot v^-) \neq 0,\\
x^-_{0,0} \cdot w \otimes v^- &= \delta(aq^{n+2-\ell}z)\psi(q^{2\ell-1-n})^{-s}(x^-_{0,0} \cdot w) \otimes v^- \neq 0.
\end{aligned}$$

Hence the vector $w \otimes v^-$ is not 0-extremal, and v is not extremal for $\mathcal{U}^h_q(sl^{tor}_{n+1})$.

$\square$

Remark 3.2.12. *Note that the obstruction to do not have extremal loop weight modules when*

$$(\ell \neq 1, n) \text{ or } (n = 2r+1 \text{ odd and } \ell \neq 1, r+1, n)$$

provides to the cyclicity of the Dynkin diagram of type $A^{(1)}_n$. For quantum affinizations associated to Dynkin diagrams without cycle, we expect to obtain extremal loop weight modules for all the nodes of the Dynkin diagram.

So we have to assume that ℓ satisfies

$$(n \text{ is even and } \ell = 1, n) \text{ or } (n = 2r+1 \text{ is odd and } \ell = 1, r+1, n). \tag{C}$$

This hypothesis occurs in Chapter 2 for the construction of the extremal fundamental loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-modules $V(e^{\varpi_\ell} Y_{\ell,a} Y_{0,aq^{d_\ell}}^{-1})$ (where we have to assume that $n = 2r+1$ is odd in addition) in the following way: its construction is based on monomial realizations of the extremal fundamental weight $\mathcal{U}_q(\hat{sl}_{n+1})$-crystals $\mathcal{B}(\varpi_\ell)$. It is possible to associate representations of $\mathcal{U}_q(sl_{n+1}^{tor})$ to these monomial crystals only if they satisfy a combinatorial condition providing to the theory of q–characters (recall that a monomial set which satisfies this condition is called a closed monomial set). It holds if and only if $\ell = 1, r+1$ or n. In these cases, the representations associated to the monomial realizations of $\mathcal{B}(\varpi_\ell)$ are the extremal fundamental loop weight modules $V(e^{\varpi_\ell} Y_{\ell,a} Y_{0,aq^{d_\ell}}^{-1})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$. Here this hypothesis reappears in a very different way for $T(e^{s\varpi_\ell} Y_{\ell,a}^s Y_{0,aq^{d_\ell}}^{-s})$ which is a priori not related to the theory of monomial crystals.

It remains to study the representations $T(e^{s\varpi_\ell} Y_{\ell,a}^s Y_{0,aq^{d_\ell}}^{-s})$ when ℓ satisfies (C). We conjecture that

Conjecture 3.2.13. *Fix $a \in \mathbb{C}^*$ and $s \in \mathbb{N}^*$ and assume that ℓ satisfies (C). Then $T(e^{s\varpi_\ell} Y_{\ell,a}^s Y_{0,aq^{d_\ell}}^{-s})$ is an extremal loop weight module of $\mathcal{U}_q(sl_{n+1}^{tor})$ generated by the vector v of ℓ-weight $e^{s\varpi_\ell} Y_{\ell,a}^s Y_{0,aq^{d_\ell}}^{-s}$. Further if we assume in addition that $s = 1$, $T(e^{\varpi_\ell} Y_{\ell,a} Y_{0,aq^{d_\ell}}^{-1})$ is isomorphic to the extremal fundamental loop weight module $V(e^{\varpi_\ell} Y_{\ell,a} Y_{0,aq^{d_\ell}}^{-1})$.*

Remark 3.2.14. *Via the coproduct Δ_D, we have also defined $\mathcal{U}_q(sl_{n+1}^{tor})$-module structures on the following tensor products (Remark 3.2.7):*

- $V_1 = V(e^{s\Lambda_\ell} Y_{\ell,aq^u}^s) \otimes V(e^{-s\Lambda_0} Y_{0,aq^{d_\ell - v}}^{-s})$ *with* $1 \leq \ell \leq n, a \in \mathbb{C}^*, s \in \mathbb{N}^*$ *and* $u, v \in \mathbb{N}$,
- $V_2 = V(e^{2\Lambda_1} Y_{1,q} Y_{1,q^{-1}}) \otimes V(e^{-2\Lambda_0} Y_{0,q^2}^{-1} Y_{0,1}^{-1})$.

When $u \neq 0$ or $v \neq 0$, the $\mathcal{U}_q(sl_{n+1}^{tor})$-module V_1 is less interesting for our study of extremal loop weight modules. In fact, let us keep the notation v^+ (resp. v^-) for the ℓ-highest weight vector (resp. the ℓ-lowest weight vector) of $V(e^{s\Lambda_\ell} Y_{\ell,aq^u}^s)$ (resp. of $V(e^{-s\Lambda_0} Y_{0,aq^{d_\ell - v}}^{-s})$). Then we check easily that if $u \neq 0$ or $v \neq 0$, the vector $v = v^+ \otimes v^-$ is not extremal for the horizontal quantum affine subagebra.

For the $\mathcal{U}_q(sl_{n+1}^{tor})$-module V_2, it is not obvious for the author to show that this tensor product gives an extremal loop weight module. As in Conjecture 3.2.13, the main difficulty is to compute the action of the quantum toroidal algebra on each component of the tensor product.

3.2.3 Proof of Conjecture 3.2.13 for $\ell = 1, n$ and $s = 1$

In this section we prove Conjecture 3.2.13 for $\ell = 1, n$ and $s = 1$. To do that, let us begin by improving the action of the quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$ on

$$T(e^{\varpi_1} Y_{1,a} Y_{0,aq}^{-1}) \subseteq V(e^{\Lambda_1} Y_{1,a}) \otimes V(e^{-\Lambda_0} Y_{0,aq}^{-1}).$$

Proposition 3.2.15. *The $\mathcal{U}_q(sl_{n+1}^{tor})$-module $T = T(e^{\varpi_1} Y_{1,a} Y_{0,aq}^{-1})$ ($a \in \mathbb{C}^*$) is thin and satisfies*

$$\begin{aligned}\chi_q(T) &= \sum_{k \in \mathbb{Z}} \Big(e^{\Lambda_1 - \Lambda_0 - k\delta} Y_{1,aq^{k(n+1)}} Y_{0,aq^{1+k(n+1)}}^{-1} + e^{\Lambda_2 - \Lambda_1 - k\delta} Y_{2,aq^{1+k(n+1)}} Y_{1,aq^{2+k(n+1)}}^{-1} \\ &\quad + \cdots + e^{\Lambda_0 - \Lambda_n - k\delta} Y_{0,aq^{n+1+k(n+1)}} Y_{n,aq^{n+k(n+1)}}^{-1} \Big).\end{aligned}$$

Further there is a basis $\{v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}}\}_{j\in\mathbb{Z}}$ *of* T *such that for all* $j \in \mathbb{Z}$,

$$\mathrm{wt}\left(v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}}\right) = \Lambda_r - \Lambda_{r-1} - k\delta \text{ with } j = k(n+1) + r, 1 \leq r \leq n+1.$$

The action on it is given by ($i \in I, j \in \mathbb{Z}$)

$$\begin{aligned}
x_i^+(z) \cdot v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}} &= \delta_{i,\overline{j-1}}\delta(aq^{j-1}z) \cdot v_{Y^{-1}_{j-2,aq^{j-1}}Y_{j-1,aq^{j-2}}},\\
x_i^-(z) \cdot v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}} &= \delta_{i,\bar{j}}\delta(aq^{j}z) \cdot v_{Y^{-1}_{j,aq^{j+1}}Y_{j+1,aq^{j}}},\\
\phi_i^\pm(z) \cdot v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}} &= \begin{cases} \psi(aq^jz) \cdot v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}} & \text{if } i = \bar{j},\\ \psi(aq^{j+1}z)^{-1} \cdot v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}} & \text{if } i = \overline{j-1},\\ v_{Y^{-1}_{j-1,aq^j}Y_{j,aq^{j-1}}} & \text{otherwise.}\end{cases}
\end{aligned}$$

The proof of Proposition 3.2.15 is given in Appendix B. As a direct consequence we have

Proposition 3.2.16. *Set* $\ell = 1$ *or* $\ell = n$. *The* $\mathcal{U}_q(sl_{n+1}^{tor})$*-module* $T(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq}^{-1})$ ($a \in \mathbb{C}^*$) *is an extremal loop weight module of* ℓ*-weight* $e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq}^{-1}$. *It is isomorphic to the extremal fundamental loop weight module* $V(e^{\varpi_\ell}Y_{\ell,a}Y_{0,aq}^{-1})$.

Proof. Assume first that $\ell = 1$. All the defining conditions of an extremal loop weight module are clearly satisfied from the formulas given in the last proposition. So the only thing we still have to prove is that $T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ and $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ are isomorphic. But the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ is explicitly known (see Theorem 2.3.12) and it is the same as the one on $T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$. The case $\ell = n$ follows by applying the automorphism ι. □

Remark 3.2.17. *Analogue representations are used in [FJMM13] for the deformed quantum toroidal algebra* $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$. *They are defined as the quantum version of a module over a Lie algebra of difference operators, and called vector representations. Actually the modules* $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ ($a \in \mathbb{C}^*$) *can be obtained from the vector representations by specialization at* $d = 1$ *(see Chapter 2). This is why we call them the specialized vector representations of* $\mathcal{U}_q(sl_{n+1}^{tor})$.

3.3 Tensor products of specialized vector representations

In this section we study tensor products of specialized vector representations $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ ($a \in \mathbb{C}^*$). We obtain in this way new families of extremal loop weight modules for $\mathcal{U}_q(sl_{n+1}^{tor})$.

The tensor product of two vector representations is also considered in [FJMM13] for the d-deformation $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$: existence conditions of the action are given. We recall these results in the first subsection for the specialized quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$.

In the second subsection we study the case of tensor products of k specialized vector representations with generic non-zero complex parameters $a_1, \cdots, a_k \in \mathbb{C}^*$ (Theorem 3.3.2): it is an irreducible extremal loop weight module of ℓ-weight

$$e^{k\varpi_1}Y_{1,a_1}\cdots Y_{1,a_k}Y_{0,a_1q}^{-1}\cdots Y_{0,a_kq}^{-1}.$$

In the third subsection we are interested in tensor products of specialized vector representations when the set of parameters forms a q-segment $\{a, aq^{-2}, \cdots, aq^{-2(k-1)}\}$ ($k \in \mathbb{N}^*$, $a \in \mathbb{C}^*$): under existence conditions it is an extremal loop weight module with an irreducible ℓ-extremal quotient (Theorem 3.3.5 and Proposition 3.3.7). When $n = 3$ and $k = 2$ this quotient is isomorphic to the $\mathcal{U}_q(sl_4^{tor})$-module $V(e^{2\varpi_1}Y_{1,a}Y_{1,aq^{-2}}Y_{0,aq}^{-1}Y_{0,aq^{-1}}^{-1})$ we defined in Section 2.4. Eventually when $n = 2r+1$ is odd and $k = r+1$, we recover the extremal fundamental loop weight modules (Theorem 3.3.13).

In the last subsection, we specialize q at roots of unity ϵ. We obtain new irreducible finite-dimensional representations of the quantum toroidal algebra $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$.

3.3.1 Existence conditions of tensor products of vector representations

Let us consider the extremal fundamental loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ ($a \in \mathbb{C}^*$), also called *specialized vector representation*. By Proposition 3.2.15, it has a basis $\{v_{Y_{j-1,aq^j}^{-1}Y_{j,aq^{j-1}}}\}_{j\in\mathbb{Z}}$. Motivated by [HN06] and the theory of crystal bases of $\mathcal{U}_q(\hat{sl}_{n+1})$-modules, we set

$$\boxed{j}_{a} = v_{Y_{j-1,aq^j}^{-1}Y_{j,aq^{j-1}}} \qquad \text{for } j \in \mathbb{Z},\ a \in \mathbb{C}^*,$$

where we still use the index convention $Y_{j+n+1,*}^{\pm 1} = Y_{j,*}^{\pm}$. In particular, we have

$$\boxed{j+n+1}_{a} = \boxed{j}_{aq^{n+1}}.$$

With these notations the $\mathcal{U}_q(sl_{n+1}^{tor})$-module structure on $V(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ is for all $i \in I$ and $j \in \mathbb{Z}$,

$$\begin{aligned}
x_i^+(z)\cdot \boxed{j}_{a} &= \delta_{i,\overline{j-1}}\delta(aq^{j-1}z)\cdot \boxed{j-1}_{a},\\
x_i^-(z)\cdot \boxed{j}_{a} &= \delta_{i,\overline{j}}\delta(aq^{j}z)\cdot \boxed{j+1}_{a},\\
\phi_i^{\pm}(z)\cdot \boxed{j}_{a} &= \begin{cases} \psi(aq^jz)\cdot \boxed{j}_{a} & \text{if } i = \overline{j},\\ \psi(aq^{j+1}z)^{-1}\cdot \boxed{j}_{a} & \text{if } i = \overline{j-1},\\ \boxed{j}_{a} & \text{otherwise.}\end{cases}
\end{aligned} \tag{3.6}$$

Consider the tensor product

$$V(\boxed{1}_{a}) \otimes V(\boxed{1}_{b}) \text{ with } a, b \in \mathbb{C}^*.$$

The following result is given in [FJMM13] for the d-deformation $\mathcal{U}_{q,d}(sl_{n+1}^{tor})$ of the quantum toroidal algebra.

Proposition 3.3.1. *[FJMM13]*

(i) The action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on the tensor product $V(\boxed{1}_{a}) \otimes V(\boxed{1}_{b})$ is well-defined if and only if $\frac{a}{b} \notin q^{(n+1)\mathbb{Z}}$.

(ii) Assume that $\frac{a}{b} = q^{-2+m(n+1)}$, $m \in \mathbb{Z}$. The module $V(\boxed{1}_{a}) \otimes V(\boxed{1}_{b})$ has a submodule spanned by vectors of the form $\boxed{i}_{a} \otimes \boxed{j}_{b}$ with $i \leq j + m(n+1)$ ($i, j \in \mathbb{Z}$). The submodule and the quotient module are irreducible and thin.

(iii) *Assume that* $\dfrac{a}{b} \in q^{2+m(n+1)}$, $m \in \mathbb{Z}$. *The module* $V(\boxed{\ 1\ }_a) \otimes V(\boxed{\ 1\ }_b)$ *has a submodule spanned by vectors of the form* $\boxed{\ i\ }_a \otimes \boxed{\ j\ }_b$ *with* $i < j + m(n+1)$ $(i, j \in \mathbb{Z})$. *The submodule and the quotient module are irreducible and thin.*

(iv) *In all the other cases,* $V(\boxed{\ 1\ }_a) \otimes V(\boxed{\ 1\ }_b)$ *is a thin, irreducible* $\mathcal{U}_q(sl_{n+1}^{tor})$*-module.*

In the following sections, we give new results about tensor products of specialized vector representations in the context of extremal loop weight modules.

3.3.2 The generic case

The main result of this section is

Theorem 3.3.2. *Let* $k \in \mathbb{N}^*$ *and* $a_1, \cdots, a_k \in \mathbb{C}^*$ *be such that*

$$\frac{a_i}{a_j} \notin q^{(n+1)\mathbb{Z}} \text{ and } \frac{a_i}{a_j} \notin q^{\pm 2+(n+1)\mathbb{Z}} \text{ for all } i < j.$$

Then the tensor product of specialized vector representations

$$V(\boxed{\ 1\ }_{a_1}) \otimes V(\boxed{\ 1\ }_{a_2}) \otimes \cdots \otimes V(\boxed{\ 1\ }_{a_k})$$

is an irreducible, thin, extremal loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$*-module of* ℓ*-weight*

$$e^{k\varpi_1} Y_{1,a_1} \cdots Y_{1,a_k} Y_{0,qa_1}^{-1} \cdots Y_{0,qa_k}^{-1}.$$

Let us begin by the following identity which will be used later.

Lemma 3.3.3. *Let* $k \in \mathbb{N}^*$. *We have the equality into the field* $\mathbb{C}(X_1, \cdots, X_k, Q)$ *of rational fractions on the formal variables* $X_1, \cdots, X_k, Q$,

$$\sum_{1 \leq i \leq k} \prod_{j \neq i} \frac{QX_i - Q^{-1}X_j}{X_i - X_j} = \frac{Q^k - Q^{-k}}{Q - Q^{-1}} = [k]_Q. \tag{3.7}$$

We would like to thank Jeremy Daniel, Dragos Fratila, Victoria Lebed, Louis-Hadrien Robert and Michal Zidor for giving us different proofs of (3.7). We give here one of them due to Michal.

Proof. Let us denote $\mathbf{X} = (X_1, \cdots, X_k)$ and $\mathbf{X_j} = (X_1, \cdots X_{j-1}, X_{j+1}, \cdots, X_k)$ for all $1 \leq j \leq k$. The well-known Vandermonde determinant is

$$V(\mathbf{X}) = \det(V_1, \cdots, V_k) = \begin{vmatrix} X_1^{k-1} & X_1^{k-2} & \cdots & 1 \\ X_2^{k-1} & X_2^{k-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ X_k^{k-1} & X_k^{k-2} & \cdots & 1 \end{vmatrix} = \prod_{1 \leq i < j \leq k} (X_i - X_j).$$

Denote by $\Sigma_i(\mathbf{X}) = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq k} X_{j_1} X_{j_2} \cdots X_{j_k}$ the i-th elementary symmetric polynomial in variables $X_1, \cdots, X_k$. By convention, we set $\Sigma_0(\mathbf{X}) = 1$.

To prove (3.7), we will show that the coefficient of degree Q^{k-1-2i} $(0 \leq i \leq k-1)$ in the left hand side is 1. By a direct computation, it is equal to

$$(-1)^i \left(\frac{X_1^{k-1-i} \Sigma_i(\mathbf{X_1})}{\prod_{j \neq 1} (X_1 - X_j)} + \cdots + \frac{X_k^{k-1-i} \Sigma_i(\mathbf{X_k})}{\prod_{j \neq k} (X_k - X_j)} \right).$$

By setting to the same denominator, we obtain

$$(-1)^i \left(\frac{X_1^{k-1-i}\Sigma_i(\mathbf{X_1})V(\mathbf{X_1}) + \cdots + (-1)^{k-1} X_k^{k-1-i}\Sigma_i(\mathbf{X_k})V(\mathbf{X_k})}{V(\mathbf{X})} \right). \tag{3.8}$$

But for all $1 \leq l \leq k$,

$$X_l^{k-1-i}\Sigma_i(\mathbf{X_l}) = X_l^{k-1-i}\Sigma_i(\mathbf{X}) - X_l^{k-1-i+1}\Sigma_{i-1}(\mathbf{X}) + \cdots + (-1)^i X_l^{k-1}\Sigma_0(\mathbf{X}).$$

Hence the numerator in (3.8) is the determinant

$$\det(V_{i+1}\Sigma_i(\mathbf{X}) - V_i\Sigma_{i-1}(\mathbf{X}) + \cdots + (-1)^i V_1\Sigma_0(\mathbf{X}), V_2, \cdots, V_k)$$

expanding along the first column. As it is equal to $(-1)^i V(\mathbf{X})$, we obtain the result. □

Let us give the proof of Theorem 3.3.2.

Proof. By Lemma 1.8.3 and Remark 1.8.4, the $\mathcal{U}_q(sl_{n+1}^{tor})$-module

$$V(\boxed{1}_{a_1}) \otimes V(\boxed{1}_{a_2}) \otimes \cdots \otimes V(\boxed{1}_{a_k})$$

is integrable and it satisfies the first and the third defining conditions of extremal loop weight modules. So it remains to prove that $\boxed{1}_{a_1} \otimes \cdots \otimes \boxed{1}_{a_k}$ is extremal for the horizontal quantum affine subalgebra. Let us show that

$$(x_{1,0}^-)^{(k)} \cdot \boxed{1}_{a_1} \otimes \cdots \otimes \boxed{1}_{a_k} = \boxed{2}_{a_1} \otimes \cdots \otimes \boxed{2}_{a_k}.$$

We give a recursive argument on k. It is satisfied if k is equal to one. Assume that $k \geq 2$ and the property true at the rank $k-1$. The vector $x_{1,0}^- \cdot \boxed{1}_{a_1} \otimes \cdots \otimes \boxed{1}_{a_k}$ is equal to

$$\boxed{1}_{a_1} \otimes \cdots \otimes \boxed{2}_{a_k} + \psi(a_k/a_{k-1})\boxed{1}_{a_1} \otimes \cdots \otimes \boxed{2}_{a_{k-1}} \otimes \boxed{1}_{a_k} + \cdots$$
$$+\psi(a_2/a_1)\psi(a_3/a_1)\cdots\psi(a_k/a_1) \cdot \boxed{2}_{a_1} \otimes \cdots \otimes \boxed{1}_{a_k}.$$

But the action of $x_{1,0}^-$ on the j-term $\boxed{1}_{a_1} \otimes \cdots \otimes \boxed{2}_{a_j} \otimes \cdots \otimes \boxed{1}_{a_k}$ reduces to the case $k-1$ for the tensor product

$$V(\boxed{1}_{a_1}) \otimes \cdots \otimes V(\boxed{1}_{a_{j-1}}) \otimes V(\boxed{1}_{a_{j+1}}) \otimes \cdots \otimes V(\boxed{1}_{a_k}).$$

By the recursive hypothesis, $(x_{1,0}^-)^{(k-1)} \cdot \boxed{1}_{a_1} \otimes \cdots \otimes \boxed{2}_{a_j} \otimes \cdots \otimes \boxed{1}_{a_k}$ is equal to

$$\psi(a_1/a_j)\psi(a_2/a_j)\cdots\psi(a_{j-1}/a_j) \cdot \boxed{2}_{a_1} \otimes \cdots \otimes \boxed{2}_{a_j} \otimes \cdots \otimes \boxed{2}_{a_k}$$

where the factor $\psi(a_1/a_j)\psi(a_2/a_j)\cdots\psi(a_{j-1}/a_j)$ stem from the box $\boxed{2}_{a_j}$. By summation we obtain

$$(x_{1,0}^-)^{(k)} \cdot \boxed{1}_{a_1} \otimes \cdots \otimes \boxed{1}_{a_k} = \frac{1}{[k]_q} \left(\sum_{1 \leq i \leq k} \prod_{j \neq i} \psi(a_j/a_i) \right) \cdot \boxed{2}_{a_1} \otimes \cdots \otimes \boxed{2}_{a_k}$$

and by (3.7) we have

$$\frac{1}{[k]_q} \left(\sum_{1 \leq i \leq k} \prod_{j \neq i} \psi(a_j/a_i) \right) = 1.$$

More generally, we show in the same way that for all $i \in I$ and $j \in \mathbb{Z}$,

$$\begin{aligned}(x^+_{j-1,0})^{(k)} \cdot \boxed{j}_{a_1} \otimes \cdots \otimes \boxed{j}_{a_k} &= \boxed{j-1}_{a_1} \otimes \cdots \otimes \boxed{j-1}_{a_k},\\ (x^-_{j,0})^{(k)} \cdot \boxed{j}_{a_1} \otimes \cdots \otimes \boxed{j}_{a_k} &= \boxed{j+1}_{a_1} \otimes \cdots \otimes \boxed{j+1}_{a_k},\\ x^\pm_{i,0} \cdot \boxed{j}_{a_1} \otimes \cdots \otimes \boxed{j}_{a_k} &= 0 \text{ otherwise.}\end{aligned}$$

Hence the vector $\boxed{1}_{a_1} \otimes \cdots \otimes \boxed{1}_{a_k}$ is extremal for the horizontal quantum affine subalgebra, and $V(\boxed{1}_{a_1}) \otimes \cdots \otimes V(\boxed{1}_{a_k})$ is an extremal loop weight module. □

3.3.3 The non-generic case

Let us consider the tensor product of specialized vector representations [2] ($k \in \mathbb{N}^*, a \in \mathbb{C}^*$)

$$V(\boxed{1}_a) \otimes V(\boxed{1}_{aq^{-2}}) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}}).$$

Denote by $\tilde{V}\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{k}_a\end{array}\right)$ the subvector space of $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$ generated by

$$\boxed{i_1}_a \otimes \boxed{i_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{i_k}_{aq^{-2(k-1)}} \text{ with } i_1 < i_2 < \cdots < i_k$$

and $V(\boxed{1}\cdots\boxed{1}_a)$ the subvector space of $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$ generated by

$$\boxed{j_1}_a \otimes \boxed{j_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{j_k}_{aq^{-2(k-1)}} \text{ with } j_1 \geq j_2 \geq \cdots \geq j_k.$$

We determine when the action of $\mathcal{U}_q(sl^{tor}_{n+1})$ is well-defined on these vector spaces.

Proposition 3.3.4. *(i) The coproduct Δ_D endows $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$ with a structure of thin $\mathcal{U}_q(sl^{tor}_{n+1})$-module if and only if*

$$(n \text{ is even and } k \leq n+1) \text{ or } \left(n \text{ is odd and } k \leq \frac{n+1}{2}\right). \tag{E}$$

(ii) The coproduct Δ_D endows $\tilde{V}\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{k}_a\end{array}\right)$ with a structure of thin $\mathcal{U}_q(sl^{tor}_{n+1})$-module if and only if k satisfies (E).

(iii) The coproduct Δ_D endows $V(\boxed{1}\cdots\boxed{1}_a)$ with a structure of thin $\mathcal{U}_q(sl^{tor}_{n+1})$-module for all $a \in \mathbb{C}^*$ and $k \in \mathbb{N}^*$.

Proof. Let us begin to show the defining conditions for these modules. Assume that $k > n+1$. By straightforward computations we check that the action of $x^-_{1,0}$ is not well-defined on the vector

$$\boxed{1}_a \otimes \boxed{2+n+1}_{aq^{-2}} \otimes \cdots \otimes \boxed{1+2(n+1)}_{aq^{-2(n+1)}} \otimes \boxed{(n+1)^{n+3}}_{aq^{-2(n+2)}} \otimes \cdots \otimes \boxed{(n+1)^k}_{aq^{-2(k-1)}}.$$

2. It could be interesting to consider the tensor product $V(\boxed{1}_a) \otimes V(\boxed{1}_{aq^2}) \otimes \cdots \otimes V(\boxed{1}_{aq^{2(k-1)}})$ instead of the one in this section. Its study is similar and it leads to the same results.

In the same way if n is odd and $k > \frac{n+1}{2} = l$, we check that the action of $x^-_{1,0}$ is not well-defined on

$$\boxed{1}_a \otimes \boxed{3}_{aq^{-2}} \otimes \cdots \otimes \boxed{1+l}_{aq^{-2(l-1)}} \otimes \boxed{1+2l}_{aq^{-2l}} \otimes \boxed{(n+1)^{l+2}}_{aq^{-2(l+1)}} \otimes \cdots \otimes \boxed{(n+1)^k}_{aq^{-2(k-1)}}.$$

It shows that (E) is needed in (i) and (ii).

Now assume that k satisfies (E). To show that the action is well-defined on the tensor product, it suffices to prove that it is well-defined on a vector

$$\boxed{i_1}_{aq^{-2(r-1)}} \otimes \boxed{i_2}_{aq^{-2(s-1)}} \in V(\boxed{1}_{aq^{-2(r-1)}}) \otimes V(\boxed{1}_{aq^{-2(s-1)}})$$

with $1 \leq r < s \leq k$ and $i_1, i_2 \in \mathbb{Z}$. This is in fact the case because $\dfrac{aq^{-2(r-1)}}{aq^{-2(s-1)}} = q^{2(s-r)}$ which is not in $q^{(n+1)\mathbb{Z}}$ by hypothesis. So (i) is proved.

Assume always that k satisfies (E) and let us prove (ii). The action of the $x_i^\pm(z)$ is well-defined on $\tilde{V}\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{k}_a\end{array}\right)$. It remains to show that for a vector

$$v = \boxed{i_1}_a \otimes \boxed{i_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{i_k}_{aq^{-2(k-1)}} \text{ with } i_1 < i_2 < \cdots < i_k,$$

$x_i^\pm(z) \cdot v \in \tilde{V}\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{k}_a\end{array}\right)$ for all $i \in I$ (Lemma 1.8.1). This is a consequence of the equalities $(i \in \mathbb{Z})$

$$\begin{array}{rcl} x^+_{\bar{i}}(z) \cdot \boxed{i}_a \otimes \boxed{i+1}_{aq^{-2}} & = & \delta(aq^{i-2}z)\psi(q^2) \cdot \boxed{i}_a \otimes \boxed{i}_{aq^{-2}} = 0, \\ x^-_{\bar{i}}(z) \cdot \boxed{i}_a \otimes \boxed{i+1}_{aq^{-2}} & = & \delta(aq^i z)\psi(1)^{-1} \cdot \boxed{i+1}_a \otimes \boxed{i+1}_{aq^{-2}} = 0. \end{array} \tag{3.9}$$

Fix $k \in \mathbb{N}^*$ and let us prove (iii). Denote by W the subvector space of the tensor product generated by vectors $\boxed{j_1}_a \otimes \boxed{j_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{j_k}_{aq^{-2(k-1)}}$ such that there exists $1 \leq s \leq k-1$ and $j_s < j_{s+1}$. We have the decomposition of vector space

$$V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}}) = V(\boxed{1}\cdots\boxed{1}_a) \oplus W$$

and $\phi_i^\pm(z) \cdot V(\boxed{1}\cdots\boxed{1}_a) \subset V(\boxed{1}\cdots\boxed{1}_a)$, $\phi_i^\pm(z) \cdot W \subset W$, $i \in I$. Let us prove that the action of $x_i^\pm(z)$ is well-defined on $V(\boxed{1}\cdots\boxed{1}_a)$ for all $i \in I$. For that, it suffices to consider a vector of the form

$$\boxed{j_r}_{aq^{-2(r-1)}} \otimes \boxed{j_s}_{aq^{-2(s-1)}} \text{ with } 1 \leq r < s \leq k \text{ and } j_r \geq j_s.$$

The action is well-defined on it, except perhaps when j_r and j_s are of the form $j_r = i + m(n+1), j_s = i$ $(i \in \mathbb{Z}, m \in \mathbb{N})$. But in this case, one has

$$\begin{aligned} x^-_{\bar{i}}(z) \cdot \boxed{j_r} \otimes \boxed{j_s} = & \delta(aq^{-2(s-1)+j_s}z) \cdot \boxed{j_r} \otimes \boxed{j_s+1} \\ & + \delta(aq^{-2(r-1)+j_r}z)\psi(q^{2(r-s)}q^{-m(n+1)}) \cdot \boxed{j_r+1} \otimes \boxed{j_s}. \end{aligned}$$

As $2(r-s) - m(n+1) < 0$, the action of $x_i^-(z)$ is well-defined. It remains to check if the last condition in Lemma 1.8.2 is satisfied. For that, the vectors we have to consider are of the form

$$v = \boxed{j_1}_a \otimes \boxed{j_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{j_k}_{aq^{-2(k-1)}} \in W$$

where there exist $1 \leq s \leq k-1$ and $i \in \mathbb{Z}$ such that $j_s = i$ and $j_{s+1} = i+1$. Then it is a direct consequence of the equality

$$x_{\bar{i}}^{-}(z) \cdot \boxed{i}_{aq^{-2(s-1)}} \otimes \boxed{i+1}_{aq^{-2s}} = 0.$$

Hence by Lemma 1.8.2, $V(\boxed{1} \cdots \boxed{1}_a)$ can be endowed with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$-module.

As $V(\boxed{1}_a)$ is thin, all the considered modules here are weighted. Actually they are thin for the following reasons: if (E) holds, all the sets $\mathcal{M}\left(V(\boxed{1}_{aq^{-2(i-1)}})\right)$ for various $1 \leq i \leq k$ are disjoint to each other and the joint spectrum of $\mathcal{U}_q(\hat{\mathfrak{h}})$ is simple. For $V(\boxed{1} \cdots \boxed{1}_a)$, the thin property is a direct consequence of its definition. Indeed for $j_1 \geq j_2 \geq \cdots \geq j_k$, the ℓ-weight of $\boxed{j_1}_a \otimes \cdots \otimes \boxed{j_k}_{aq^{-2(k-1)}}$ is

$$m = Y_{\bar{j_1},aq^{j_1-1}} Y_{\overline{j_1-1},aq^{j_1}}^{-1} Y_{\bar{j_2},aq^{j_2-3}} Y_{\overline{j_2-1},aq^{j_2-2}}^{-1} \cdots Y_{\bar{j_k},aq^{j_k-2(k-1)}} Y_{\overline{j_k-1},aq^{j_k-2(k-1)}}^{-1}.$$

And such an ℓ-weight determines completely the sequence $j_1 \geq j_2 \geq \cdots \geq j_k$. Then we have a bijective correspondence between vectors $\boxed{j_1}_a \otimes \cdots \otimes \boxed{j_k}_{aq^{-2(k-1)}}$ with $j_1 \geq j_2 \geq \cdots \geq j_k$ and their ℓ-weight, and $V(\boxed{1} \cdots \boxed{1}_a)$ is thin. □

Theorem 3.3.5. *Assume that*

$$(n \text{ is even and } k < n+1) \text{ or } \left(n \text{ is odd and } k < \frac{n+1}{2}\right). \tag{E'}$$

Then the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$ is an extremal loop weight module of ℓ-weight

$$e^{k\varpi_1} Y_{1,a} Y_{1,aq^{-2}} \cdots Y_{1,aq^{-2(k-1)}} Y_{0,aq}^{-1} Y_{0,aq^{-1}}^{-1} \cdots Y_{0,aq^{-2(k-1)+1}}^{-1}.$$

Proof. By Lemma 1.8.3 and Remark 1.8.4, $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$ is an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module which satisfies the third condition of Definition 1.7.1.

Let us show the extremality of

$$\boxed{1}_a \otimes \boxed{1}_{aq^{-2}} \otimes \cdots \otimes \boxed{1}_{aq^{-2(k-1)}}$$

for the horizontal quantum affine subalgebra: the vector $x_{1,0}^{-} \cdot \boxed{1}_a \otimes \cdots \otimes \boxed{1}_{aq^{-2(k-1)}}$ is equal to

$$\boxed{1} \otimes \cdots \otimes \boxed{1} \otimes \boxed{2} + \psi(q^{-2}) \cdot \boxed{1} \otimes \cdots \otimes \boxed{2} \otimes \boxed{1} + \cdots$$

$$+ \psi(q^{-2})\psi(q^{-4}) \cdots \psi(q^{-2(k-1)}) \cdot \boxed{2} \otimes \cdots \otimes \boxed{1} \otimes \boxed{1}$$

By (3.9) the only vector we have to keep is the last one, and we take its image by the operator $x_{1,0}^{-}$. We repeat this operation k times. Finally we obtain

$$(x_{1,0}^{-})^k \cdot \boxed{1} \otimes \cdots \otimes \boxed{1} = \psi(q^{-2})^{k-1} \psi(q^{-4})^{k-2} \cdots \psi(q^{-2(k-1)}) \cdot \boxed{2} \otimes \cdots \otimes \boxed{2}.$$

But $\psi(q^{-2(j-1)}) = \frac{[j]_q}{[j-1]_q}$ for all $j \geq 2$, and

$$\psi(q^{-2})^{k-1} \psi(q^{-4})^{k-2} \cdots \psi(q^{-2(k-1)}) = [k]_q!.$$

Hence $(x^-_{1,0})^{(k)} \cdot \boxed{1} \otimes \cdots \otimes \boxed{1} = \boxed{2} \otimes \cdots \otimes \boxed{2}$. More generally we show that for all $i \in I$ and $j \in \mathbb{Z}$,

$$\begin{aligned}
(x^+_{j-1,0})^{(k)} \cdot \boxed{j}_a \otimes \cdots \otimes \boxed{j}_{aq^{-2(k-1)}} &= \boxed{j-1}_a \otimes \cdots \otimes \boxed{j-1}_{aq^{-2(k-1)}}, \\
(x^-_{j,0})^{(k)} \cdot \boxed{j}_a \otimes \cdots \otimes \boxed{j}_{aq^{-2(k-1)}} &= \boxed{j+1}_a \otimes \cdots \otimes \boxed{j+1}_{aq^{-2(k-1)}}, \\
x^\pm_{i,0} \cdot \boxed{j}_a \otimes \cdots \otimes \boxed{j}_{aq^{-2(k-1)}} &= 0 \text{ otherwise.}
\end{aligned}$$

So the vector $\boxed{1}_a \otimes \cdots \otimes \boxed{1}_{aq^{-2(k-1)}}$ is extremal for the horizontal quantum affine subalgebra.

It remains to show that the vector

$$\boxed{1}_a \otimes \boxed{1}_{aq^{-2}} \otimes \cdots \otimes \boxed{1}_{aq^{-2(k-1)}}$$

generates this module when k satisfies (E'). It reduces to prove it for all tensor product of the form

$$V(\boxed{1}_{aq^{-2(r-1)}}) \otimes V(\boxed{1}_{aq^{-2(s-1)}}) \text{ with } 1 \leq r < s \leq k.$$

So let $\boxed{j_1}_{aq^{-2(r-1)}} \otimes \boxed{j_2}_{aq^{-2(s-1)}}$ be a vector in this tensor product. By the computations done above, one can assume that $j_1, j_2 \geq 1$. We show easily that it can be obtained from $\boxed{1}_{aq^{-2(r-1)}} \otimes \boxed{1}_{aq^{-2(s-1)}}$ by the following straightforward calculations ($u \in \mathbb{Z}$)

$$\begin{aligned}
x^-_1(z) \cdot \boxed{1} \otimes \boxed{1+u(n+1)} &= \delta(aq^{-2(s-1)+1+u(n+1)}z) \cdot \boxed{1} \otimes \boxed{2+u(n+1)} \\
&\quad + \delta(aq^{-2(r-1)+1}z)\psi(q^{2(r-s)+u(n+1)}) \cdot \boxed{2} \otimes \boxed{1+u(n+1)}, \\
x^-_1(z) \cdot \boxed{2} \otimes \boxed{1+u(n+1)} &= \delta(aq^{-2(s-1)+1}z) \cdot \boxed{2} \otimes \boxed{2+u(n+1)}.
\end{aligned}$$

Let us note that all the coefficients in these equalities are non-zeros when k satisfies (E'). Then we obtain the result in that case. $\square$

Remark 3.3.6. *Assume that*

$$(n \text{ is even and } k = n+1) \text{ or } \left(n \text{ is odd and } k = \frac{n+1}{2}\right).$$

and set

$$V = \mathcal{U}_q(sl^{tor}_{n+1}) \cdot \boxed{1}_a \otimes \boxed{1}_{aq^{-2}} \otimes \cdots \otimes \boxed{1}_{aq^{-2(k-1)}}.$$

Then V is an extremal loop weight module, and it is strictly included in $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$.

Proposition 3.3.7. *The $\mathcal{U}_q(sl^{tor}_{n+1})$-module $V(\boxed{1} \cdots \boxed{1}_a)$ ($a \in \mathbb{C}^*, k \in \mathbb{N}^*$) is an irreducible extremal loop weight module of ℓ-weight*

$$e^{k\varpi_1} Y_{1,a} Y_{1,aq^{-2}} \cdots Y_{1,aq^{-2(k-1)}} Y^{-1}_{0,aq} Y^{-1}_{0,aq^{-1}} \cdots Y^{-1}_{0,aq^{-2(k-1)+1}}.$$

Proof. The proof is analogue to the one of Theorem 3.3.5, and we only show the irreducibility of $V(\boxed{1} \cdots \boxed{1}_a)$. As above, it follows from the case of the tensor product ($r \geq 2$)

$$V(\boxed{1}_a) \otimes V(\boxed{1}_{aq^{-2(r-1)}})$$

and the calculations ($u \in \mathbb{N}$)

$$\begin{aligned}
x_1^+(z)\cdot \boxed{2}\otimes\boxed{2-u(n+1)} &= \delta(aqz)\cdot\boxed{1}\otimes\boxed{2-u(n+1)} + \delta(aq^{-2(r-1)+1-u(n+1)}z)\times\\
&\quad \psi(q^{2+2(r-1)+u(n+1)})^{-1}\cdot\boxed{2}\otimes\boxed{1-u(n+1)}\\
x_1^+(z)\cdot \boxed{2}\otimes\boxed{1-u(n+1)} &= \delta(aqz)\cdot\boxed{1}\otimes\boxed{1-u(n+1)}.\\
x_1^-(z)\cdot \boxed{1}\otimes\boxed{1-u(n+1)} &= \delta(aq^{1-2(r-1)-u(n+1)}z)\cdot\boxed{1}\otimes\boxed{2-u(n+1)}\\
&\quad +\delta(aqz)\psi(q^{-2(r-1)-u(n+1)})\cdot\boxed{2}\otimes\boxed{1-u(n+1)}\\
x_1^-(z)\cdot \boxed{2}\otimes\boxed{1-u(n+1)} &= \delta(aq^{1-2(r-1)-u(n+1)}z)\cdot\boxed{2}\otimes\boxed{2-u(n+1)}.
\end{aligned}$$

In particular we have in $V(\boxed{1}\boxed{1}_a)$

$$\begin{aligned}
x_1^+(z)\cdot\boxed{2}\otimes\boxed{2} &= \delta(aq^{-1}z)\psi(q^4)^{-1}\cdot\boxed{2}\otimes\boxed{1},\\
x_1^-(z)\cdot\boxed{1}\otimes\boxed{1} &= \delta(aqz)\psi(q^{-2})\cdot\boxed{2}\otimes\boxed{1}.
\end{aligned}$$

Note also that all the coefficients in these equalities are non-zeros. □

In Section 2.4 we have defined an irreducible extremal loop weight module $V(e^{2\varpi_1}Y_{1,a}Y_{1,aq^{-2}}Y_{0,aq}^{-1}Y_{0,aq^{-1}}^{-1})$ ($a\in\mathbb{C}^*$) of extremal ℓ-weight $e^{2\varpi_1}Y_{1,a}Y_{1,aq^{-2}}Y_{0,aq}^{-1}Y_{0,aq^{-1}}^{-1}$ for the quantum toroidal algebra $\mathcal{U}_q(sl_4^{tor})$. For that we have considered a monomial realization of the extremal crystal $\mathcal{B}(2\varpi_1)$. The tensor product construction gives a new way to obtain this module.

Proposition 3.3.8. *The $\mathcal{U}_q(sl_4^{tor})$-modules $V(e^{2\varpi_1}Y_{1,a}Y_{1,aq^{-2}}Y_{0,aq}^{-1}Y_{0,aq^{-1}}^{-1})$ and $V(\boxed{1}\boxed{1}_a)$ are isomorphic.*

Proof. Let us prove this result when $a=1$ (the other cases follow by twisting the action by the automorphism t_a). The action of $\mathcal{U}_q(sl_4^{tor})$ on $V(e^{2\varpi_1}Y_{1,q}Y_{1,q^{-1}}Y_{0,q^2}^{-1}Y_{0,1}^{-1})$ is known (see Theorem 2.4.6). Let us give a basis of $V(\boxed{1}\boxed{1}_1)$ on which the action writes in the same way. The representation $V(\boxed{1}\boxed{1}_1)$ is thin and has a basis formed by the ℓ-weight vectors

$$v_m=\boxed{j_1}_1\otimes\boxed{j_2}_{q^{-2}} \text{ with } j_1\geq j_2$$

and where the monomial m is the ℓ-weight of $\boxed{j_1}_1\otimes\boxed{j_2}_{q^{-2}}$. We normalize this basis as follows: for $s\in\mathbb{N}^*$ and $k\in\mathbb{Z}$ we set

$$w_m:=\psi(q^{4s+2})^{-1}\cdot v_m$$

if $m=Y_{1+k,q^{1+4s+k}}Y_{k,q^{2+4s+k}}^{-1}Y_{2+k,q^k}Y_{1+k,q^{k+1}}^{-1}$. If $m=Y_{2+k,q^{2+k}}Y_{1+k,q^{3+k}}^{-1}Y_{1+k,q^{-1+k}}Y_{k,q^k}^{-1}$ we define the vector

$$w_m:=\psi(q^{-2})^{-1}\cdot v_m.$$

For the other ℓ-weights m, we just set $w_m:=v_m$. The action of the quantum toroidal algebra $\mathcal{U}_q(sl_4^{tor})$ can be computed on the basis $(w_m)_m$. And we check that the formulas of the action are the same as the explicit ones for the module $V(e^{2\varpi_1}Y_{1,q}Y_{1,q^{-1}}Y_{0,q^2}^{-1}Y_{0,1}^{-1})$ given in Theorem 2.4.6. □

Remark 3.3.9. *Consider the tensor product ($k\in\mathbb{N}^*$, $a_1,\cdots,a_k\in\mathbb{C}^*$)*

$$V=V(e^{\varpi_n}Y_{n,a_1}Y_{0,a_1q}^{-1})\otimes V(e^{\varpi_n}Y_{n,a_2}Y_{0,a_2q}^{-1})\otimes\cdots\otimes V(e^{\varpi_n}Y_{n,a_k}Y_{0,a_kq}^{-1}).$$

By using the automorphism ι of $\mathcal{U}_q(sl_{n+1}^{tor})$, we obtain from the previous results that

– *if* $\frac{a_i}{a_j} \notin q^{(n+1)\mathbb{Z}}$ *or* $\frac{a_i}{a_j} \notin q^{\pm 2+(n+1)\mathbb{Z}}$ *for all* $i < j$*,* V *is an irreducible, thin, extremal loop weight* $\mathcal{U}_q(sl_{n+1}^{tor})$*-module of* ℓ*-weight*

$$e^{k\varpi_n} Y_{n,a_1} Y_{n,a_2} \cdots Y_{n,a_k} Y_{0,a_1q}^{-1} Y_{0,a_2q}^{-1} \cdots Y_{0,a_kq}^{-1},$$

– *if* k *satisfies* (E') *and* $a_i = aq^{-2(i-1)}$ *for all* $1 \le i \le k$ $(a \in \mathbb{C}^*)$*,* V *is an extremal loop weight* $\mathcal{U}_q(sl_{n+1}^{tor})$*-module of* ℓ*-weight*

$$e^{k\varpi_n} Y_{n,a} Y_{n,aq^{-2}} \cdots Y_{n,aq^{-2(k-1)}} Y_{0,aq}^{-1} Y_{0,aq^{-1}}^{-1} \cdots Y_{0,aq^{-2(k-1)+1}}^{-1}.$$

Furthermore for all $k \in \mathbb{N}^*$*, there exists a subvector space of* V *which can be endowed with a structure of irreducible, thin, extremal loop weight* $\mathcal{U}_q(sl_{n+1}^{tor})$*-module of* ℓ*-weight*

$$e^{k\varpi_n} Y_{n,a} Y_{n,aq^{-2}} \cdots Y_{n,aq^{-2(k-1)}} Y_{0,aq}^{-1} Y_{0,aq^{-1}}^{-1} \cdots Y_{0,aq^{-2(k-1)+1}}^{-1}.$$

Proposition 3.3.10. *Assume that* k *satisfies* (E')*. Then* $\tilde{V} = \tilde{V}\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{k}_{\,a}\end{array}\right)$ *is an irreducible* $\mathcal{U}_q(sl_{n+1}^{tor})$*-module. Furthermore, the vector* $v = \boxed{1}_a \otimes \cdots \otimes \boxed{k}_{aq^{-2(k-1)}}$ *is not extremal for the horizontal quantum affine subalgebra in that case.*

Proof. As above, the irreducibility of $\tilde{V}$ follows from the case of the tensor product

$$V(\boxed{1}_{aq^{-2(r-1)}}) \otimes V(\boxed{1}_{aq^{-2(s-1)}}) \text{ with } 1 \le r < s \le k$$

and the calculations $(u \in \mathbb{N}, v \in \mathbb{N}^*)$

$$\begin{aligned}
x_1^+(z) \cdot \boxed{2} \otimes \boxed{2+v(n+1)} &= \delta(aq^{-2(r-1)+1}z) \cdot \boxed{1} \otimes \boxed{2+v(n+1)} + \psi(q^{2+2(s-r)-v(n+1)})^{-1} \\
&\quad \times \delta(aq^{-2(s-1)+1+v(n+1)}z) \cdot \boxed{2} \otimes \boxed{1+v(n+1)}, \\
x_1^+(z) \cdot \boxed{1} \otimes \boxed{2+u(n+1)} &= \begin{cases} 0 \text{ if } s = r+1 \text{ and } u = 0, \\ \psi(q^{2(s-r)-u(n+1)}) \times \\ \delta(aq^{-2(s-1)+1+u(n+1)}z) \cdot \boxed{1} \otimes \boxed{1+u(n+1)} & \text{otherwise.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
x_1^-(z) \cdot \boxed{1} \otimes \boxed{1+v(n+1)} &= \delta(aq^{-2(s-1)+1+v(n+1)}z) \cdot \boxed{1} \otimes \boxed{2+v(n+1)} \\
&\quad + \delta(aq^{-2(r-1)+1}z)\psi(q^{2(r-s)+v(n+1)}) \cdot \boxed{2} \otimes \boxed{1+v(n+1)}, \\
x_1^-(z) \cdot \boxed{1} \otimes \boxed{2+u(n+1)} &= \begin{cases} 0 \text{ if } s = r+1 \text{ and } u = 0, \\ \delta(aq^{-2(r-1)+1}z) \times \\ \psi(q^{2(s-r)+2+u(n+1)})^{-1} \cdot \boxed{2} \otimes \boxed{2+u(n+1)} & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that all the coefficients in these equalities are non-zeros when k satisfies (E').

Let us show that the vector

$$v = \boxed{1}_a \otimes \boxed{2}_{aq^{-2}} \otimes \cdots \otimes \boxed{k}_{aq^{-2(k-1)}}$$

is not extremal for the horizontal quantum affine subalgebra. We apply operators $x_{k,0}^-$, $x_{k+1,0}^-, \cdots, x_{n,0}^-$ on it: it is equal to

$$\boxed{1}_a \otimes \cdots \otimes \boxed{k-1}_{aq^{-2(k-2)}} \otimes \boxed{n+1}_{aq^{-2(k-1)}} = \boxed{1}_a \otimes \cdots \otimes \boxed{k-1}_{aq^{-2(k-2)}} \otimes \boxed{0}_{aq^{-2(k-1)+n+1}}.$$

The weight λ of this vector satisfies $\lambda(h_0) = 0$ and we have

$$x^+_{0,0} \cdot \boxed{1}_a \otimes \cdots \otimes \boxed{0}_{aq^{-2(k-1)+n+1}} = \boxed{0}_a \otimes \boxed{2}_{aq^{-2}} \cdots \otimes \boxed{0}_{aq^{-2(k-1)+n+1}},$$
$$x^-_{0,0} \cdot \boxed{1}_a \otimes \cdots \otimes \boxed{0}_{aq^{-2(k-1)+n+1}} = \boxed{1}_a \otimes \cdots \otimes \boxed{k-1}_{aq^{-2(k-2)}} \otimes \boxed{1}_{aq^{-2(k-1)+n+1}}.$$

Hence this vector is not 0-extremal, and v is not extremal for the horizontal quantum affine subalgebra. So $\tilde{V}$ is not an extremal loop weight module of ℓ-weight $e^{\varpi_k} Y_{k,aq^{-k+1}} Y^{-1}_{0,aq}$ when k satisfies (E'). $\square$

Assume that $n = 2r+1$ ($r \in \mathbb{N}^*$) is odd and $k = r+1$. Denote by $V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{matrix}\right)$ (resp. W) the subvector space of $\tilde{V}\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{matrix}\right)$ generated by

$$\boxed{i_1}_a \otimes \boxed{i_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{i_{r+1}}_{aq^{-2r}} \text{ with } i_1 < i_2 < \cdots < i_{r+1}$$

and $i_{r+1} - i_1 \leq n$ (resp. $i_{r+1} - i_1 > n$).

Proposition 3.3.11. *The coproduct Δ_D endows $V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{matrix}\right)$ and W with a structure of irreducible and thin $\mathcal{U}_q(sl^{tor}_{n+1})$-module.*

Proof. Let us begin to prove that W can be endowed with a structure of $\mathcal{U}_q(sl^{tor}_{n+1})$-module. By Lemma 1.8.1, it suffices to prove that for all vector of the form $v = \boxed{i_1}_a \otimes \boxed{i_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{i_{r+1}}_{aq^{-2r}}$ with $i_1 < i_2 < \cdots < i_{r+1}$ and $i_{r+1} - i_1 > n$, $x^\pm_i \cdot v \in W$. This is a consequence of the following equalities in $V(\boxed{1}_a) \otimes V(\boxed{1}_{aq^{-2r}})$

$$x^+_1(z) \cdot \boxed{2}_a \otimes \boxed{2+(n+1)}_{aq^{-2r}} = x^+_1(z) \cdot \boxed{2}_a \otimes \boxed{2}_{aq^2} = \delta(aqz) \boxed{1}_a \otimes \boxed{2}_{aq^2},$$
$$x^-_1(z) \cdot \boxed{1}_a \otimes \boxed{1+(n+1)}_{aq^{-2r}} = x^-_1(z) \cdot \boxed{1}_a \otimes \boxed{1}_{aq^2} = \delta(aq^3 z) \boxed{1}_a \otimes \boxed{2}_{aq^2}.$$

The action on $V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{matrix}\right)$ can be defined as the quotient of $\tilde{V}\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{matrix}\right)$ by its submodule W. We check directly the irreducibility of these modules from the formulas given in the proof of Proposition 3.3.10. $\square$

For all Young tableau $T = (1 \leq i_1 < i_2 < \cdots < i_{r+1} \leq n+1)$ of shape $(r+1)$, $a \in \mathbb{C}^*$ and $s \in \mathbb{Z}$, we set

$$T_{aq^{2s}} = \boxed{i_s + \left[\tfrac{s-1}{r+1}\right](n+1)}_a \otimes \cdots \otimes \boxed{i_{r+s} + \left[\tfrac{r+s-1}{r+1}\right](n+1)}_{aq^{-2r}}$$

where $i_{k+r+1} = i_k$ by convention. Then $\{T_{aq^{2s}}\}_{T,s}$ is a basis of $V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{matrix}\right)$. We determine the action on it.

Proposition 3.3.12. *The action of* $\mathcal{U}_q(sl_{n+1}^{tor})$ *on* $V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right)$ *is given by*

$$
\begin{aligned}
x_i^+(z)\cdot T_b &= \sum_{1\le p\le r+1} \delta_{\{i_p=i+1\}}\delta_{\{i_{p-1}\neq i\}}\delta(bq^{i-2p}z)(\tilde{e}_i\cdot T)_{bq^{-2\delta_{i,0}}},\\
x_i^-(z)\cdot T_b &= \sum_{1\le p\le r+1} \delta_{\{i_p=i\}}\delta_{\{i_{p+1}\neq i+1\}}\delta(bq^{i-2p}z)(\tilde{f}_i\cdot T)_{bq^{2\delta_{i,0}}},
\end{aligned}
\tag{3.10}
$$

$$
\phi_i^\pm(z)\cdot T_b = \begin{cases} \psi(bq^{i-2p+2}z)^{-1}T_b & \text{if } \exists p \text{ s.t. } i_p=i+1, i_{p-1}\neq i,\\ \psi(bq^{i-2p}z)T_b & \text{if } \exists p \text{ s.t. } i_p=i, i_{p+1}\neq i+1,\\ T_b & \text{otherwise,}\end{cases}
\tag{3.11}
$$

where $T=(1\le i_1<i_2<\cdots<i_{r+1}\le n+1)$ *is a Young tableau of shape* $(r+1)$ *and* $i\in I, b\in aq^{2\mathbb{Z}}$.

Here we use the notation $\delta_A=1$ if A is true and $\delta_A=0$ otherwise. Let us describe the operators $\tilde{e}_i$ and $\tilde{f}_i$ occurring in (3.10). For that denote by $\mathcal{B}_0(\Lambda_{r+1})'_{\mathrm{aff}}$ the set of Young tableaux of shape $(r+1)$ (we use the same notations as in Chapter 2). It can be endowed with an affine P_{cl}-crystal structure as follows: the weight function is given by the content of tableaux, i.e. $\mathrm{wt}(T):=(w_1(T),\cdots,w_{n+1}(T))$ where $w_i(T)$ is the number of letters i occurring in the tableau T. It can be viewed as an element of P_{cl} by setting

$$\mathrm{wt}(T)=\mathrm{cl}\left(w_1(T)(\Lambda_1-\Lambda_0)+w_2(T)(\Lambda_2-\Lambda_1)+\cdots+w_{n+1}(T)(\Lambda_0-\Lambda_n)\right).$$

The operators $\tilde{e}_i$ and $\tilde{f}_i$ in (3.10) are the Kashiwara operators in this crystal. Their action on the Young tableaux of shape $(r+1)$ is such that: for $i\neq 0$ we have $\tilde{e}_i\cdot T=T'$ or 0 where T' is obtained from T by replacing $i+1$ by i. If it is not possible (i.e. when we have both $i+1$ and i in T or when $i+1$ does not occur in T), then it is zero. Similarly $\tilde{f}_i\cdot T=T''$ or 0, where T'' is given by replacing i by $i+1$. For the action of $\tilde{e}_0$ and $\tilde{f}_0$, we have

$$
\tilde{e}_0\cdot T=\begin{cases}0 & \text{if } i_1\neq 1 \text{ or } i_{r+1}=n+1,\\ (i_2,\cdots,i_{r+1},n+1) & \text{if } i_1=1 \text{ and } i_{r+1}\neq n+1,\end{cases}
$$

$$
\tilde{f}_0\cdot T=\begin{cases}0 & \text{if } i_1=1 \text{ or } i_{r+1}\neq n+1,\\ (1,i_1,\cdots,i_r) & \text{if } i_1\neq 1 \text{ and } i_{r+1}=n+1.\end{cases}
$$

Proof. Formulas (3.10) and (3.11) follows directly from the previous computations. □

Let us recall some facts about the extremal fundamental loop weight modules (see also Chapter 2). By construction $V(e^{\varpi_{r+1}}Y_{r+1,a}Y^{-1}_{0,aq^{r+1}})$ has a basis parametrized by a monomial realization of the level 0 extremal weight crystal $\mathcal{B}(\varpi_{r+1})$. This crystal is related to the affine P_{cl}-crystal $\mathcal{B}_0(\Lambda_{r+1})'_{\mathrm{aff}}$ in the following way: from $\mathcal{B}_0(\Lambda_{r+1})'_{\mathrm{aff}}$ one can define its affinization $\mathcal{B}_0(\Lambda_{r+1})_{\mathrm{aff}}$ (see [Kas02b, Man12c]): this is the P-crystal with vertices in $\{z^s\otimes T|s\in\mathbb{Z}, T\in\mathcal{B}_0(\Lambda_{r+1})'_{\mathrm{aff}}\}$ such that for all $s\in\mathbb{Z}$ and $T\in\mathcal{B}_0(\Lambda_{r+1})'_{\mathrm{aff}}$,

$$
\begin{gathered}
\mathrm{wt}(z^s\otimes T)=\mathrm{wt}(T)+s\delta,\\
\tilde{e}_i\cdot(z^s\otimes T)=z^{s+\delta_{i,0}}\otimes(\tilde{e}_i\cdot T),\ \tilde{f}_i\cdot(z^s\otimes T)=z^{s-\delta_{i,0}}\otimes(\tilde{f}_i\cdot T).
\end{gathered}
$$

Then it is known [Kas02b] that $\mathcal{B}_0(\Lambda_{r+1})_{\mathrm{aff}} \simeq \mathcal{B}(\varpi_{r+1})$. In particular, one can also parametrize the monomial basis of $V(e^{\varpi_{r+1}}Y_{r+1,a}Y^{-1}_{0,aq^{r+1}})$ by $\mathcal{B}_0(\Lambda_{r+1})_{\mathrm{aff}}$ (a couple (T,s) of a Young tableau $T = (1 \leq i_1 < \cdots < i_{r+1})$ of shape $(r+1)$ and an integer $s \in \mathbb{Z}$), by setting

$$T_{aq^{2s}} = \prod_{1\leq j\leq r+1} \boxed{i_j}_{\,aq^{2s}}.$$

Note that the coefficients q^{2s} in the indices of the basis $\{T_{aq^{2s}}\}_{T,s}$ stem from the monomial realization of $\mathcal{B}(\varpi_{r+1})$: we have the equality of $\mathcal{U}_q(sl_{n+1})$-crystals

$$\mathcal{B}(\varpi_{r+1}) = \bigsqcup_{s\in\mathbb{Z}} z^s \otimes \mathcal{B}_0(\Lambda_{r+1})$$

and for all $s \in \mathbb{Z}$ the monomial realization of $z^s \otimes \mathcal{B}_0(\Lambda_{r+1})$ is $\mathcal{M}(Y_{r+1,q^{-2s}}Y^{-1}_{0,q^{r+1-2s}})$ (see Chapter 2).

The similarities with the $\mathcal{U}_q(sl^{tor}_{n+1})$-module $V\begin{pmatrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_{\,a}\end{pmatrix}$ we defined in this section suggest the following result.

Theorem 3.3.13. *Assume that $n = 2r+1$ $(r \in \mathbb{N}^*)$. Then $V = V\begin{pmatrix}\boxed{1}\\ \vdots \\ \boxed{r+1}_{\,a}\end{pmatrix}$ is an irreducible extremal loop weight module of ℓ-weight $e^{\varpi_{r+1}}Y_{r+1,aq^{-r-2}}Y^{-1}_{0,aq^{-1}}$. Furthermore, it is isomorphic to the extremal fundamental loop weight module $V(e^{\varpi_{r+1}}Y_{r+1,aq^{-r-2}}Y^{-1}_{0,aq^{-1}})$.*

Proof. The fact that V is isomorphic to $V(e^{\varpi_{r+1}}Y_{r+1,aq^{-r-2}}Y^{-1}_{0,aq^{-1}})$ is straightforward from the explicit formulas of the action of $\mathcal{U}_q(sl^{tor}_{n+1})$ on $V(e^{\varpi_{r+1}}Y_{r+1,aq^{-r-2}}Y^{-1}_{0,aq^{-1}})$ given in Section 2.3.2. The result follows directly. □

Remark 3.3.14. *Assume that n is even. The $\mathcal{U}_q(sl^{tor}_{n+1})$-module $\tilde{V}\begin{pmatrix}\boxed{1}\\ \vdots \\ \boxed{n+1}_{\,a}\end{pmatrix}$ is not irreducible: in fact let us denote by $V\begin{pmatrix}\boxed{1}\\ \vdots \\ \boxed{n+1}_{\,a}\end{pmatrix}$ (resp. W) its subvector space generated by*

$$\boxed{i_1}_{\,a} \otimes \boxed{i_2}_{\,aq^{-2}} \otimes \cdots \otimes \boxed{i_{n+1}}_{\,aq^{-2n}} \text{ with } i_1 < i_2 < \cdots < i_{n+1}$$

and $i_{n+1} - i_1 \leq 2n+1$ (resp. $i_{n+1} - i_1 > 2n+1$). We show that the coproduct Δ_D endows these vector spaces with a structure of irreducible and thin $\mathcal{U}_q(sl^{tor}_{n+1})$-module. But $V\begin{pmatrix}\boxed{1}\\ \vdots \\ \boxed{n+1}_{\,a}\end{pmatrix}$ is not an extremal loop weight module, the vector

$$\boxed{1}_{\,a} \otimes \boxed{2}_{\,aq^{-2}} \otimes \cdots \otimes \boxed{n+1}_{\,aq^{-2n}}$$

being not extremal for the horizontal quantum affine subalgebra.

3.3.4 Finite-dimensional representations at roots of unity

Set $L \geq 1$ and let ϵ be a primitive $[(n+1)L]$-root of unity. Recall that $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$ is the algebra defined as $\mathcal{U}_q(sl^{tor}_{n+1})$ with ϵ instead of q (without divided power and derivation element). For all $a \in \mathbb{C}^*$, let $V(\boxed{1}_a)_\epsilon$ be the $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$-module constructed in Section 2.3.3. It is obtained from $V(\boxed{1}_a)$ by specializing q at ϵ in (3.6) and by taking a quotient. Furthermore, $V(\boxed{1}_a)_\epsilon$ is an irreducible module of dimension $[(n+1)L]$.

Theorem 3.3.15. *Assume that ϵ is a primitive $[(n+1)L]$-root of unity and let $k \in \mathbb{N}^*$ and $a_1, \cdots, a_k \in \mathbb{C}^*$ be such that $\frac{a_i}{a_j} \notin \epsilon^{(n+1)\mathbb{Z}}$ and $\frac{a_i}{a_j} \notin \epsilon^{\pm 2+(n+1)\mathbb{Z}}$ for all $1 \leq i, j \leq k$. Then Δ_D endows the tensor product*

$$V(\boxed{1}_{a_1})_\epsilon \otimes V(\boxed{1}_{a_2})_\epsilon \otimes \cdots \otimes V(\boxed{1}_{a_k})_\epsilon$$

with a well-defined structure of $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$-module. Furthermore it is irreducible and finite-dimensional of dimension $[(n+1)L]^k$.

Proof. It is straightforward from the action of $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$ on $V(\boxed{1}_a)_\epsilon$ and the computations done above. □

Let us consider the tensor product $(k \in \mathbb{N}^*, a \in \mathbb{C}^*)$

$$V(\boxed{1}_a)_\epsilon \otimes V(\boxed{1}_{aq^{-2}})_\epsilon \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})_\epsilon.$$

We determine when the action of $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$ is well-defined on it.

Theorem 3.3.16. *The coproduct Δ_D endows the tensor product*

$$V(\boxed{1}_a)_\epsilon \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})_\epsilon$$

with a structure of $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$-module if and only if k satisfies (E). Furthermore this module is irreducible and finite-dimensional of dimension $[(n+1)L]^k$.

Proof. The proof is analogue to the one when q is not a root of unity. Further we show by straightforward computations that $V(\boxed{1}_a)_\epsilon \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})_\epsilon$ is irreducible. □

Remark 3.3.17. *Let us consider the $\mathcal{U}_q(sl^{tor}_{n+1})$-modules $\tilde{V}\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{k}_a\end{array}\right)$ and $V(\boxed{1}\cdots\boxed{1}_a)$ and their specialization at q to ϵ: one can show that they are well-defined if and only if k satisfies (E). Furthermore the $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$-modules hence obtained are equal to the tensor product $V(\boxed{1}_a)_\epsilon \otimes V(\boxed{1}_{aq^{-2}})_\epsilon \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})_\epsilon$ defined above.*

3.4 Tensor products of $V(e^{\varpi_{r+1}}Y_{r+1,a}Y^{-1}_{0,aq^{r+1}})$ for $\mathcal{U}_q(sl^{tor}_{2r+2})$

In this section we use the same process as in Section 3.3: we consider tensor products of fundamental extremal loop weight modules $V(e^{\varpi_\ell}Y_{\ell,a}Y^{-1}_{0,aq^\ell})$ when $n = 2r+1$ is odd and $\ell = r+1$ here. We obtain new extremal loop weight modules for $\mathcal{U}_q(sl^{tor}_{n+1})$.

So in all this section $n = 2r+1$ ($r \geq 1$) is an odd integer and $\ell = r+1$. In the first subsection we recall the formulas of the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right)$ and we determine when the action of the quantum toroidal algebra is well-defined on a tensor product of these extremal fundamental loop weight modules (Proposition 3.4.1).

In the second subsection, we study the case of tensor products with generic complex parameters $a_1, \cdots, a_k \in \mathbb{C}^*$ ($k \in \mathbb{N}^*$): it is an irreducible, thin, extremal loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module of ℓ-weight

$$e^{k\varpi_{r+1}} Y_{r+1,a_1} \cdots Y_{r+1,a_k} Y_{0,a_1 q^{r+1}}^{-1} \cdots Y_{0,a_k q^{r+1}}^{-1} \text{ (Theorem 3.4.3).}$$

In the last subsection we specialize q at roots of unity ϵ. We get in this way new irreducible finite-dimensional modules over $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$.

3.4.1 Existence conditions of tensor products

In Section 3.3.3, we gave a basis $\{T_{aq^{2s}}\}_{T,s}$ of the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right)$ indexed by the Young tableaux of shape $(r+1)$ $T = (1 \leq i_1 < i_2 < \cdots < i_{r+1} \leq n+1)$ and integers $s \in \mathbb{Z}$. Let us recall the formulas of the action on it ($i \in I, b \in aq^{2\mathbb{Z}}$)

$$\begin{aligned}
x_i^+(z) \cdot T_b &= \sum_{1 \leq p \leq r+1} \delta_{\{i_p = i+1\}} \delta_{\{i_{p-1} \neq i\}} \delta(bq^{i-2p} z) (\tilde{e}_i \cdot T)_{bq^{-2\delta_{i,0}}}, \\
x_i^-(z) \cdot T_b &= \sum_{1 \leq p \leq r+1} \delta_{\{i_p = i\}} \delta_{\{i_{p+1} \neq i+1\}} \delta(bq^{i-2p} z) (\tilde{f}_i \cdot T)_{bq^{2\delta_{i,0}}}, \\
\phi_i^\pm(z) \cdot T_b &= \begin{cases} \psi(bq^{i-2p+2} z)^{-1} T_b & \text{if } \exists p \text{ s.t. } i_p = i+1, i_{p-1} \neq i, \\ \psi(bq^{i-2p} z) T_b & \text{if } \exists p \text{ s.t. } i_p = i, i_{p+1} \neq i+1, \\ T_b & \text{otherwise,} \end{cases}
\end{aligned}$$

where $\delta_A = 1$ if A is true and $\delta_A = 0$ otherwise, and $\tilde{e}_i$ and $\tilde{f}_i$ are the Kashiwara operators in the P_{cl}-crystal $\mathcal{B}_0(\Lambda_{r+1})'_{\mathrm{aff}}$.

Proof. With the notations used in Chapter 2, we set for all Young tableau T of shape $(r+1)$ and $s \in \mathbb{Z}$,

$$T_{aq^{2s}} = m_{T;s}.$$

Then (3.10) and (3.11) deduce by straightforward computations from the formulas of the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V(Y_{r+1,a} Y_{0,aq^{r+1}}^{-1})$ we gave in Chapter 2. $\square$

Let us consider the tensor product $V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right) \otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_b\end{array}\right)$.

Proposition 3.4.1. *The action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on the tensor product*

$$V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right) \otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_b\end{array}\right)$$

is well-defined if and only if $\frac{a}{b}\notin q^{2\mathbb{Z}}$. *In this case, the* $\mathcal{U}_q(sl^{tor}_{n+1})$*-module hence obtained is irreducible and thin.*

Proof. Let us consider

$$T^{(1)}=(1\leq i_1^{(1)}<\cdots<i_\ell^{(1)}\leq n+1) \text{ and } T^{(2)}=(1\leq i_1^{(2)}<\cdots<i_\ell^{(2)}\leq n+1)$$

two Young tableaux of shape $(r+1)$ and $\alpha\in aq^{2\mathbb{Z}},\beta\in bq^{2\mathbb{Z}}$. We study the action of $x_i^+(z)$ on $T^{(1)}_\alpha\otimes T^{(2)}_\beta$:

$$\begin{aligned}x_i^+(z)\cdot T^{(1)}_\alpha\otimes T^{(2)}_\beta = &\sum_{1\leq u\leq r+1}\delta_{\{i_u^{(1)}=i+1\}}\delta(\alpha q^{i-2u}z)(\tilde{e}_i\cdot T^{(1)})_{\alpha q^{-2\delta_{i,0}}}\otimes T^{(2)}_\beta+\\ \sum_{u,v}\Bigg[1+\delta_{\{i_u^{(1)}=i+1,i_{u-1}^{(1)}\neq i\}}&\left(\psi\left(\frac{\alpha}{\beta}q^{2(v-u)+2}\right)^{-1}-1\right)+\delta_{\{i_u^{(1)}=i,i_{u+1}^{(1)}\neq i+1\}}\left(\psi\left(\frac{\alpha}{\beta}q^{2(v-u)}\right)-1\right)\Bigg]\\ &\times\delta_{\{i_v^{(2)}=i+1\}}\delta(\beta q^{i-2v}z)T^{(1)}_\alpha\otimes(\tilde{e}_i\cdot T^{(2)})_{\beta q^{-2\delta_{i,0}}}.\end{aligned}$$

Hence the coefficients of the series $x_i^+(z)\cdot T^{(1)}_\alpha\otimes T^{(2)}_\beta$ are well-defined on the tensor product if and only if for all $\alpha\in aq^{2\mathbb{Z}}$, $\beta\in bq^{2\mathbb{Z}}$ and $1\leq u,v\leq r+1$,

$$\frac{\alpha}{\beta}q^{2(v-u)}\neq 1.$$

A necessary and sufficient condition is that $\frac{a}{b}\notin q^{2\mathbb{Z}}$. Similar computations for $x_i^-(z)$ lead to the same condition. Then we have the result by Lemma 1.8.1.

From now assume that $\frac{a}{b}\notin q^{2\mathbb{Z}}$ and let us show that the module hence obtained is thin and irreducible. Recall that the extremal loop weight module $V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_a\end{array}\right)$ is irreducible, thin and the indices of the variables $Y_{i,*}^{\pm 1}$ occurring in its q–character are in $aq^{i+1+2\mathbb{Z}}$ for all $i\in I$. As we have supposed that $\frac{a}{b}\notin q^{2\mathbb{Z}}$, the sets $aq^{i+1+2\mathbb{Z}}$ and $bq^{i+1+2\mathbb{Z}}$ are disjoints for all $i\in I$. Then $V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_a\end{array}\right)\otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_b\end{array}\right)$ is thin. It implies directly its irreducibility. □

Remark 3.4.2. *Assume that* $\frac{a}{b}\in q^{2\mathbb{Z}}$. *It would be interesting to find a subspace of* $V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_a\end{array}\right)\otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_b\end{array}\right)$ *on which the action is well-defined. But it is not obvious for us to determine the existence of such a subspace.*

3.4.2 The generic case

Theorem 3.4.3. *Let* $k\in\mathbb{N}^*$ *and* $a_1,\cdots,a_k\in\mathbb{C}^*$ *be such that* $\frac{a_i}{a_j}\notin q^{2\mathbb{Z}}$ *for all* $i<j$. *Then the tensor product*

$$V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_{a_1}\end{array}\right)\otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_{a_2}\end{array}\right)\otimes\cdots\otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots\\ \boxed{r+1}_{a_k}\end{array}\right)$$

is an irreducible, thin, extremal loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$*-module of* ℓ*-weight*

$$e^{k\varpi_{r+1}} Y_{r+1,a_1} \cdots Y_{r+1,a_k} Y_{0,a_1 q^{r+1}}^{-1} \cdots Y_{0,a_k q^{r+1}}^{-1}.$$

Proof. By Lemma 1.8.3 and Remark 1.8.4, this tensor product is an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-module which satisfies the first and the third conditions in Definition 1.7.1. Then the only thing we have to show is the extremality of

$$v = \begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_{a_1}\end{array} \otimes \cdots \otimes \begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_{a_k}\end{array}$$

for the horizontal quantum affine subalgebra. Fix $i \in I$ and let $T = (1 \leq i_1 < \cdots < i_p = i < \cdots < i_{r+1} \leq n+1)$ be a Young tableaux of shape $(r+1)$ and $T' = \tilde{f}_i \cdot T$. Let us show that

$$(x_{i,0}^-)^{(k)} \cdot T_{a_1} \otimes \cdots \otimes T_{a_k} = T'_{a_1} \otimes \cdots \otimes T'_{a_k}.$$

We have

$$\begin{aligned} x_{i,0}^- \cdot T_{a_1} \otimes \cdots \otimes T_{a_k} \quad &= \quad T_{a_1} \otimes \cdots \otimes T'_{a_k} + \psi(a_k/a_{k-1}) T_{a_1} \otimes \cdots \otimes T'_{a_{k-1}} \otimes T_{a_k} + \cdots \\ &\quad + \psi(a_2/a_1)\psi(a_3/a_1) \cdots \psi(a_k/a_1) T'_{a_1} \otimes T_{a_2} \otimes \cdots \otimes T_{a_k}. \end{aligned}$$

As in the proof of Theorem 3.3.2, we are reduced to the case $k-1$. By induction on k and using (3.7) we obtain

$$(x_{i,0}^-)^{(k)} \cdot T_{a_1} \otimes \cdots \otimes T_{a_k} = T'_{a_1} \otimes \cdots \otimes T'_{a_k}.$$

In the same way one can show that

$$(x_{i,0}^+)^{(k)} \cdot T_{a_1} \otimes \cdots \otimes T_{a_k} = T''_{a_1} \otimes \cdots \otimes T''_{a_k}$$

with $T'' = \tilde{e}_i \cdot T$. The extremality of v follows. □

3.4.3 Finite-dimensional representations at roots of unity

Set $L > 1$ and let ϵ be a primitive $[2L]$-root of unity. For all $a \in \mathbb{C}^*$, let $V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right)_\epsilon$ be the $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$-module constructed in Section 2.3.3. It is obtained from $V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right)$ by specializing q at ϵ in (3.10), (3.11) and by taking a quotient. We have proved that $V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_a\end{array}\right)_\epsilon$ is an irreducible and finite-dimensional $\mathcal{U}_\epsilon(sl_{n+1}^{tor})'$-module of dimension $L \cdot \begin{pmatrix} n+1 \\ r+1 \end{pmatrix}$.

Theorem 3.4.4. *Assume that ϵ is a primitive $[2L]$-root of unity and let $k \in \mathbb{N}^*$ and $a_1, \cdots, a_k \in \mathbb{C}^*$ be such that $\frac{a_i}{a_j} \notin \epsilon^{2\mathbb{Z}}$ for all $i < j$. Then Δ_D endows the tensor product*

$$V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_{a_1}\end{array}\right)_\epsilon \otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_{a_2}\end{array}\right)_\epsilon \otimes \cdots \otimes V\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{r+1}_{a_k}\end{array}\right)_\epsilon$$

with a well-defined structure of $\mathcal{U}_\epsilon(sl^{tor}_{n+1})'$-module. Furthermore it is irreducible and finite-dimensional of dimension $\left[L \cdot \begin{pmatrix} n+1 \\ r+1 \end{pmatrix}\right]^k$.

Proof. This is straightforward from the computations done above. $\square$

Chapter 4

Extremal loop weight modules for $\mathcal{U}_q(\hat{sl}_\infty)$

This chapter corresponds to the paper [Man13b], in preparation.

4.1 Motivations

4.1.1 Abstract

We construct by fusion product new families of irreducible representations of the quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$. The action is defined via the Drinfeld coproduct and is related to the crystal structure of semi-standard tableaux of type A_∞. We call these representations extremal loop weight modules. The main motivations are applications to quantum toroidal algebras $\mathcal{U}_q(sl_{n+1}^{tor})$: we prove the conjectural link between $\mathcal{U}_q(\hat{sl}_\infty)$ and $\mathcal{U}_q(sl_{n+1}^{tor})$ stated in [Her11] for these families of representations. We recover in this way the extremal loop weight modules obtained in [Man13a].

4.1.2 Motivations

The representation theory of the algebra $\mathcal{U}_q(\hat{sl}_\infty)$ is related to the one of quantum toroidal algebras $\mathcal{U}_q(sl_{n+1}^{tor})$ in the following way (see [Her11]) : let us consider the morphism between the corresponding Dynkin diagrams of the algebras $\mathcal{U}_q(\hat{sl}_\infty)$ and $\mathcal{U}_q(sl_{n+1}^{tor})$

$$\phi_n : i \in \mathbb{Z} \mapsto \bar{i} \in \mathbb{Z}/(n+1)\mathbb{Z}.$$

It gives rise to a ring morphism

$$\phi_n : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in\mathbb{Z},a\in\mathbb{C}^*} \longrightarrow \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I_n,a\in\mathbb{C}^*}.$$

Then the following combinatorial link between the algebras $\mathcal{U}_q(\hat{sl}_\infty)$ and $\mathcal{U}_q(sl_{n+1}^{tor})$ is expected in [Her11].

Conjecture. *[Her11, Conjecture 5.3] Let V be a simple $\mathcal{U}_q(\hat{sl}_\infty)$-module in the category $\mathcal{O}_{\mathrm{int}}$. Then the image of the q–character of V by ϕ_n is also the q–character of a representation of $\mathcal{U}_q(sl_{n+1}^{tor})$.*

The aim in [Her11] is to predict q–character formulae for representations of $\mathcal{U}_q(sl^{tor}_{n+1})$ from the representation theory of $\mathcal{U}_q(\hat{sl}_\infty)$ (which is better understood). Furthermore this conjecture is proved for the class of Kirillov-Reshetikhin modules of $\mathcal{U}_q(sl^{tor}_{n+1})$.

In this chapter, we intend to construct extremal loop weight modules of the quantum toroidal algebras $\mathcal{U}_q(sl^{tor}_{n+1})$ by using the conjectural link expected in [Her11]. More precisely our aim is twofold: first construct extremal loop weight $\mathcal{U}_q(\hat{sl}_\infty)$-modules by using the technical features introduced in Chapter 2 and Chapter 3. Second prove the conjecture above for the extremal fundamental loop weight $\mathcal{U}_q(\hat{sl}_\infty)$-modules, and see if the associated $\mathcal{U}_q(sl^{tor}_{n+1})$-modules are ℓ-extremal.

Let us explain another motivation to follow this byroad through the algebra $\mathcal{U}_q(\hat{sl}_\infty)$ for the construction of extremal loop weight modules for quantum toroidal algebras. Recall that $\mathcal{U}_q(\hat{sl}_\infty)$ is defined as limit of the quantum affine algebras $\mathcal{U}_q(\hat{sl}_{n+1})$ ($n \geq 2$). For this reason the representation theory of this algebra is better understood than for general quantum affinizations: the fundamental modules we have to consider are thin, with basis labelled by semi-standard tableaux on which the action is explicitly known. It does not hold in the quantum toroidal case: in fact for $\mathcal{U}_q(sl^{tor}_{n+1})$ the fundamental modules are not thin (see [Her11, Section 4.1]). This is one of the main reasons of the met difficulties in the study of extremal loop weight modules for the quantum toroidal algebras (besides see Remark 3.2.9). We will see that more precise results can be obtained in our construction of extremal loop weight modules for the quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$ than in the quantum toroidal case.

4.1.3 Summary of results

We construct the extremal loop weight modules on the quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$ by fusion product, in the spirit of Chapter 3. We consider the tensor product ($\ell \geq 1$ and $a \in \mathbb{C}^*$)

$$V^\infty(Y_{\ell,a}) \otimes V^\infty(Y^{-1}_{0,aq^{d_\ell}})$$

of the simple ℓ-highest weight module $V^\infty(Y_{\ell,a})$ and the simple ℓ-lowest weight module $V^\infty(Y^{-1}_{0,aq^{d_\ell}})$. We define an action of $\mathcal{U}_q(\hat{sl}_\infty)$ on this tensor product by using the Drinfeld coproduct Δ_D.

Proposition. *(Proposition 4.3.2) The coproduct Δ_D is well-defined on*

$$V^\infty(Y_{\ell,a}) \otimes V^\infty(Y^{-1}_{0,aq^{d_\ell}})$$

and endows this tensor product with a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module.

Let us note that the extremal fundamental loop weight $\mathcal{U}_q(\hat{sl}_\infty)$-modules can also be obtained from monomial crystals, in the spirit of Chapter 2. The representation $V^\infty_{1,a}$ is called the *vector representation* of $\mathcal{U}_q(\hat{sl}_\infty)$.

As in the quantum toroidal case, we consider $V^\infty_{\ell,a}$ the sub-$\mathcal{U}_q(\hat{sl}_\infty)$-module of

$$V^\infty(Y_{\ell,a}) \otimes V^\infty(Y^{-1}_{0,aq^{d_\ell}})$$

generated by $v^+ \otimes v^-$, tensor product of a ℓ-highest weight vector v^+ of $V^\infty(Y_{\ell,a})$ and a ℓ-lowest weight vector v^- of $V^\infty(Y^{-1}_{0,aq^{d_\ell}})$. We determine a basis of $V^\infty_{\ell,a}$ labelled by semi-standard tableaux $\mathcal{T}_{[1,\ell]}$ of shape (ℓ) on which the action of $\mathcal{U}_q(\hat{sl}_\infty)$ is explicitly known.

Theorem. *(Theorem 4.3.10) The action of $\mathcal{U}_q(\hat{sl}_\infty)$ on $V^\infty_{\ell,a}$ is given for all $T = (i_1 < i_2 < \cdots < i_\ell) \in \mathcal{T}_{[1,\ell]}$ by*

$$\begin{aligned}
x_i^+(z) \cdot T_a &= \sum_{1 \le p \le \ell} \delta_{\{i_p = i+1\}} \delta(aq^{i+\ell+1-2p} z)(\tilde{e}_i \cdot T)_a, \\
x_i^-(z) \cdot T_a &= \sum_{1 \le p \le \ell} \delta_{\{i_p = i\}} \delta(aq^{i+\ell+1-2p} z)(\tilde{f}_i \cdot T)_a, \\
\phi_i^\pm(z) \cdot T_a &= \begin{cases} \psi(aq^{i+\ell+3-2p} z)^{-1} \cdot T_a & \text{if there exists } 1 \le p \le \ell \text{ such that } i_p = i+1 \text{ and } i_{p-1} \neq i, \\ \psi(aq^{i+\ell+1-2p} z) \cdot T_a & \text{if there exists } 1 \le p \le \ell \text{ such that if } i_p = i \text{ and } i_{p+1} \neq i+1, \\ T_a & \text{otherwise.} \end{cases}
\end{aligned}$$

where $\tilde{e}_i, \tilde{f}_i$ are the Kashiwara operators on the $\mathcal{U}_q(sl_\infty)$-crystal $\mathcal{T}_{[1,\ell]}$.

In particular, we get the following result.

Theorem. *(Theorem 4.3.14) Set $\ell \ge 1$ and $a \in \mathbb{C}^*$. The $\mathcal{U}_q(\hat{sl}_\infty)$-module $V^\infty_{\ell,a}$ is irreducible, thin and ℓ-extremal of ℓ-weight $Y_{\ell,a} Y^{-1}_{0,aq^{d_\ell}}$. It is called the extremal fundamental loop weight module.*

We study the fusion product of extremal fundamental loop weight modules.

Proposition. *(Proposition 4.4.7) Fix $a \in \mathbb{C}^*, k \in \mathbb{N}^*$ and $\ell_1, \ell_2, \cdots, \ell_k \ge 1$. Then Δ_D endows the tensor product*

$$V^\infty_{\ell_1, aq^{-\ell_1}} \otimes V^\infty_{\ell_2, aq^{-\ell_2 - 2\ell_1}} \otimes \cdots \otimes V^\infty_{\ell_s, aq^{-\ell_s - 2(\ell_1 + \cdots + \ell_{s-1})}} \otimes \cdots \otimes V^\infty_{\ell_k, aq^{-\ell_k - 2(\ell_1 + \cdots + \ell_{k-1})}}$$

with a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module. Furthermore, it has a submodule isomorphic to

$$V^\infty_{\ell, aq^{-\ell}} \text{ with } \ell = \ell_1 + \cdots \ell_k.$$

In particular, all the extremal fundamental loop weight modules can be obtained as sub-$\mathcal{U}_q(\hat{sl}_\infty)$-modules of a fusion product of vector representations.

Let us consider the tensor product ($\ell \ge 1, a \in \mathbb{C}^*$)

$$V^\infty_{\ell,a} \otimes V^\infty_{\ell,aq^2} \otimes \cdots \otimes V^\infty_{\ell,aq^{2(k-1)}}.$$

Proposition. *(Proposition 4.4.8) There exists a subvector space $V^\infty_{\ell,\mathbf{a}}$ (with $\mathbf{a} = \{a, aq^2, \cdots, aq^{2(k-1)}\}$) of this tensor product which can be endowed with a thin $\mathcal{U}_q(\hat{sl}_\infty)$-module structure. Furthermore $V^\infty_{\ell,\mathbf{a}}$ has a basis labelled by the set of semi-standard tableaux of shape $(\ell \times k)$ on which the action of $\mathcal{U}_q(\hat{sl}_\infty)$ is explicitly known.*

Theorem. *(Theorem 4.4.9) The $\mathcal{U}_q(\hat{sl}_\infty)$-module $V^\infty_{\ell,\mathbf{a}}$ is an irreducible extremal loop weight module of ℓ-weight*

$$Y_{\ell,a} Y_{\ell,aq^2} \cdots Y_{\ell,aq^{2(k-1)}} Y^{-1}_{0,aq^\ell} Y^{-1}_{0,aq^{\ell+2}} \cdots Y^{-1}_{0,aq^{\ell+2(k-1)}}.$$

The main motivations of the study of extremal loop weight modules over $\mathcal{U}_q(\hat{sl}_\infty)$ are applications to quantum toroidal algebras $\mathcal{U}_q(sl^{tor}_{n+1})$. We have

Theorem. *(Theorem 4.5.3) Assume that $n \geq 2$ is such that*

$$(n \text{ is even and } \ell \leq n+1) \text{ or } \left(n \text{ is odd and } \ell \leq \frac{n+1}{2}\right). \tag{4.1}$$

Then $\phi_n \left(\chi_q(V(Y_{\ell,a}Y_{0,aq^\ell}^{-1}))\right)$ is the q–character of a representation of $\mathcal{U}_q(sl_{n+1}^{tor})$. We denote it $[V_{\ell,a}^\infty]_n$.

This result confirms the conjectural link expected in [Her11, Conjecture 5.3] in the context of extremal fundamental loop weight modules of $\mathcal{U}_q(\hat{sl}_\infty)$.

Recall that our aim is to construct extremal loop weight modules for the quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$. The main result in this way is the following.

Theorem. *(Theorem 4.5.4)*

(i) Assume that $n \geq 2$. Then $[V_{1,a}^\infty]_n$ is an extremal loop weight module of ℓ-weight $Y_{1,a}Y_{0,aq}^{-1}$.

(ii) Assume that $n = 2r+1$ is odd ($r \geq 1$). Then $[V_{r+1,a}^\infty]_n$ admits an irreducible quotient which is an extremal loop weight module of ℓ-weight $Y_{r+1,a}Y_{0,aq^{r+1}}^{-1}$.

We recover in this way the extremal fundamental loop weight modules of $\mathcal{U}_q(sl_{n+1}^{tor})$ constructed in Chapter 2 and Chapter 3.

4.1.4 Organization

This chapter is organized as follows.

In Section 4.2 we recall the main definitions of the quantum group $\mathcal{U}_q(sl_\infty)$ and its quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$ and we briefly review their representation theory. In particular one defines the extremal weight modules and we introduce the notion of extremal loop weight modules for $\mathcal{U}_q(\hat{sl}_\infty)$.

In Section 4.3 we construct the fusion product of fundamental ℓ-highest weight modules and fundamental ℓ-lowest weight modules associated respectively to the weights Λ_ℓ and $-\Lambda_0$ ($\ell \geq 1$) and to a non-zero complex parameter. We define a $\mathcal{U}_q(\hat{sl}_\infty)$-module structure on it by using the Drinfeld coproduct (Proposition 4.3.2). We obtain (as submodule of the fusion product) an extremal loop weight module associated to the weight $\Lambda_\ell - \Lambda_0$ (Theorem 4.3.14), called the extremal fundamental loop weight module of weight $\Lambda_\ell - \Lambda_0$. Furthermore we explicit the action of $\mathcal{U}_q(\hat{sl}_\infty)$ on it (Theorem 4.3.10).

Section 4.4 is devoted to the study of fusion products of extremal fundamental loop weight modules. When the non-zero complex parameters are chosen generic, we get extremal loop weight modules (Theorem 4.4.3). In the non-generic case we recover all the extremal fundamental loop weight modules from the fusion product of the extremal fundamental loop weight modules associated to the weight $\Lambda_1 - \Lambda_0$ and to a q-segment (Theorem 4.4.6). By fusion product of k extremal fundamental loop weight modules of weight $\Lambda_\ell - \Lambda_0$, we also obtain extremal loop weight modules of the weights $k\Lambda_\ell - k\Lambda_0$ ($\ell, k \geq 1$) (Theorem 4.4.9).

In Section 4.5, we discuss the applications to quantum toroidal algebras. We show that the q–character formulae obtained from the extremal fundamental loop weight $\mathcal{U}_q(\hat{sl}_\infty)$-modules are the q–character of representations of the quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$ (Theorem 4.5.3). Furthermore we determine when these representations give extremal loop weight modules (Theorem 4.5.4).

4.2 Background

We recall the main definitions and general properties about the representation theory of the quantum group $\mathcal{U}_q(sl_\infty)$ and its quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$.

4.2.1 Cartan matrix

Let $C = (C_{i,j})_{i,j\in\mathbb{Z}}$ be the Cartan matrix of type A_∞, that is

$$C_{i,i} = 2\ ,\ C_{i,i+1} = C_{i+1,i} = -1 \text{ and } C_{i,j} = 0 \text{ otherwise.}$$

Consider the vector space

$$\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathbb{Q}h_i$$

and the linear functions Λ_i (the fundamental weights) on $\mathfrak{h}$ given by

$$\Lambda_i(h_j) = \delta_{i,j} \text{ for all } i,j \in \mathbb{Z}.$$

For $i \in \mathbb{Z}$, set $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}$. Denote by $\Pi = \{\alpha_i, i \in \mathbb{Z}\}$ the set of simple roots and $\Pi^\vee = \{h_i, i \in \mathbb{Z}\}$ the set of simple coroots. Let $P = \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\Lambda_i$ be the weight lattice and $P^+ = \bigoplus_{i\in\mathbb{Z}} \mathbb{N}\Lambda_i$ the semigroup of dominant weights. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subseteq P$ (the root lattice) and $Q^+ = \bigoplus_{i\in I} \mathbb{N}\alpha_i \subseteq Q$. For $\lambda, \mu \in \mathfrak{h}^*$, write $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Denote by W the Weyl group of type sl_∞: it is the subgroup of transformations stabilizing P generated by the simple reflections s_i defined by $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ $(i \in \mathbb{Z}, \lambda \in P)$.

4.2.2 Quantum group $\mathcal{U}_q(sl_\infty)$

In this article $q = e^t \in \mathbb{C}^*$ $(t \in \mathbb{C})$ is not a root of unity and is fixed. For $l \in \mathbb{Z}, r \geq 0$, $m \geq m' \geq 0$ we set

$$[l]_q = \frac{q^l - q^{-l}}{q - q^{-1}} \in \mathbb{Z}[q^{\pm 1}],\ [r]_q! = [r]_q[r-1]_q \dots [1]_q,\ \begin{bmatrix} m \\ m' \end{bmatrix}_q = \frac{[m]_q!}{[m-m']_q![m']_q!}.$$

Definition 4.2.1. *The quantum group $\mathcal{U}_q(sl_\infty)$ is the $\mathbb{C}$-algebra with generators k_h $(h \in \mathfrak{h})$, $x_i^\pm$ $(i \in \mathbb{Z})$ and relations*

$$k_h k_{h'} = k_{h+h'},\ k_0 = 1,$$

$$k_h x_j^\pm k_{-h} = q^{\pm\alpha_j(h)} x_j^\pm,$$

$$[x_i^+, x_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$(x_i^\pm)^{(2)} x_{i+1}^\pm - x_i^\pm x_{i+1}^\pm x_i^\pm + x_{i+1}^\pm (x_i^\pm)^{(2)} = 0,$$

where we use the notations $k_i^\pm = k_{\pm h_i}$ and for all $r \geq 0$, $(x_i^\pm)^{(r)} = \frac{(x_i^\pm)^r}{[r]_q!}$.

The quantum group $\mathcal{U}_q(sl_\infty)$ can be considered as limit of quantum groups of type A_n and has been studied by various authors (see for example [AJL12, EK08, FM02a, LS91] and references therein).

For $J = [a,b] \subset \mathbb{Z}$ denote by $\mathcal{U}_q(sl_\infty)_J$ the subalgebra of $\mathcal{U}_q(sl_\infty)$ generated by the $x_i^\pm, k_h$ for $i \in J$, $h \in \oplus_{j\in[a,b]}\mathbb{Q}h_j$. Then $\mathcal{U}_q(sl_\infty)_J$ is isomorphic to the quantum group $\mathcal{U}_q(sl_{b-a+2})$.

Let $\mathcal{U}_q(sl_\infty)^+$ (resp. $\mathcal{U}_q(sl_\infty)^-$, $\mathcal{U}_q(\mathfrak{h})$) be the subalgebra of $\mathcal{U}_q(sl_\infty)$ generated by the x_i^+ (resp. the x_i^-, resp the k_h). We have a triangular decomposition of $\mathcal{U}_q(sl_\infty)$ (see [Lus93]):

Theorem 4.2.2. *We have an isomorphism of vector spaces*

$$\mathcal{U}_q(sl_\infty) \simeq \mathcal{U}_q(sl_\infty)^- \otimes \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q(sl_\infty)^+.$$

4.2.3 Representations of $\mathcal{U}_q(sl_\infty)$

For V a representation of $\mathcal{U}_q(sl_\infty)$ and $\nu \in P$, the weight space V_ν of V is

$$V_\nu = \{v \in V | k_i \cdot v = q^{\nu(h_i)} v, \forall i \in \mathbb{Z}\}.$$

Definition 4.2.3. *A representation V is said to be in the category $\mathcal{O}$ if*

(i) it admits a weight space decomposition $V = \bigoplus_{\nu \in P} V_\nu$,

(ii) V_ν is finite-dimensional for all ν,

(iii) $\{\nu | V_\nu \neq \{0\}\} \subset \cup_{j=1,\cdots,N}\{\nu|\nu \leq \lambda_j\}$ for some $\lambda_1, \cdots, \lambda_N \in P$.

For $\lambda \in P$, a representation V is said to be of highest weight λ if there is $v \in V_\lambda$ such that for all $i \in I, x_i^+ \cdot v = 0$ and $\mathcal{U}_q(sl_\infty) \cdot v = V$. Such a representation is in the category $\mathcal{O}$. Furthermore there is a unique simple highest weight module of highest weight λ.

Definition 4.2.4. *A representation V is said to be integrable if*

(i) it admits a weight space decomposition $V = \bigoplus_{\nu \in P} V_\nu$,

(ii) all the $x_i^\pm$ ($i \in \mathbb{Z}$) are locally nilpotent.

Theorem 4.2.5. *[Lus93, Her11] The simple highest weight module of highest weight λ is integrable if and only if λ is dominant. It is denoted $V(\lambda)$.*

Extremal weight representations

Now we recall the definition and some properties of extremal weight modules for the quantum group $\mathcal{U}_q(sl_\infty)$ [Kas94].

Definition 4.2.6. *For an integrable $\mathcal{U}_q(sl_\infty)$-module V and $\lambda \in P$, a vector $v \in V_\lambda$ is called i-extremal if $x_i^+ \cdot v = 0$ or $x_i^- \cdot v = 0$. In this case we set $S_i(v) = (x_i^-)^{(\lambda(h_i))} \cdot v$ or $S_i(v) = (x_i^+)^{(-\lambda(h_i))} \cdot v$. A vector $v \in V_\lambda$ is called extremal of weight λ if, for any $l \geq 1$, $S_{i_1} \cdots S_{i_l}(v)$ is i-extremal for any $i, i_1, \cdots, i_l \in \mathbb{Z}$.*

For v an extremal vector of weight λ, we set $W \cdot v = \{S_{i_1} \cdots S_{i_l}(v) | l \in \mathbb{N}, i_1, \cdots i_l \in \mathbb{Z}\}$.

Definition 4.2.7. *For $\lambda \in P$, the extremal weight module of extremal weight λ is the $\mathcal{U}_q(sl_\infty)$-module generated by a vector v_λ with the defining relations that v_λ is extremal of weight λ. It is denoted $V(\lambda)$.*

Example 4.2.8. *If λ is dominant, $V(\lambda)$ is the simple highest weight module of highest weight λ.*

Theorem 4.2.9. *[Kas94] For $\lambda \in P$, the module $V(\lambda)$ is integrable and has a crystal basis $\mathcal{B}(\lambda)$.*

4.2.4 Quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$

We recall the definition of the quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$ (without central charge) in terms of currents.

Definition 4.2.10. *The quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$ is the $\mathbb{C}$-algebra with generators $x_{i,r}^{\pm}$ $(i \in \mathbb{Z}, r \in \mathbb{Z})$, k_h $(h \in \mathfrak{h})$, $h_{i,m}$ $(i \in \mathbb{Z}, m \in \mathbb{Z} - \{0\})$ and the following defining relations $(i, j \in \mathbb{Z}, h, h' \in \mathfrak{h})$:*

$$k_h k_{h'} = k_{h+h'},\ k_0 = 1,$$

$$\phi_i^{\pm}(z)\phi_j^{\pm}(w) = \phi_j^{\pm}(w)\phi_i^{\pm}(z)\ ,\ \phi_i^{-}(z)\phi_j^{+}(w) = \phi_j^{+}(w)\phi_i^{-}(z),$$

$$(w - q^{\pm C_{ij}} z)\phi_i^{+}(z)x_j^{\pm}(w) = (q^{\pm C_{ij}} w - z)x_j^{\pm}(w)\phi_i^{+}(z), \tag{4.2}$$

$$(w - q^{\pm C_{ij}} z)\phi_i^{-}(z)x_j^{\pm}(w) = (q^{\pm C_{ij}} w - z)x_j^{\pm}(w)\phi_i^{-}(z), \tag{4.3}$$

$$[x_i^{+}(z), x_j^{-}(w)] = \frac{\delta_{i,j}}{q - q^{-1}}(\delta(\frac{w}{z})\phi_i^{+}(w) - \delta(\frac{z}{w})\phi_i^{-}(z)),$$

$$(z - q^{\pm C_{ij}} w)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\pm a_{ij}} z - w)x_j^{\pm}(w)x_i^{\pm}(z),$$

$$\begin{gathered} x_i^{\pm}(z_1)x_i^{\pm}(z_2)x_{i\pm1}^{\pm}(w) - (q + q^{-1})x_i^{\pm}(z_1)x_{i\pm1}^{\pm}(w)x_i^{\pm}(z_2) \\ + x_{i\pm1}^{\pm}(w)x_i^{\pm}(z_1)x_i^{\pm}(z_2) + (z_1 \leftrightarrow z_2) = 0, \end{gathered}$$

and $[x_i^{\pm}(z), x_j^{\pm}(w)] = 0$ for $i \neq j, j \pm 1$.

Here we use the formal power series $\delta(z) = \sum_{s\in\mathbb{Z}} z^s$ and

$$x_i^{\pm}(z) = \sum_{r\in\mathbb{Z}} x_{i,r}^{\pm} z^r,$$

$$\phi_i^{\pm}(z) = \sum_{m\geq 0} \phi_{i,\pm m}^{\pm} z^{\pm m} = k_i^{\pm 1}\exp(\pm(q - q^{-1})\sum_{m'\geq 1} h_{i,\pm m'} z^{\pm m'}).$$

Subalegras of $\mathcal{U}_q(\hat{sl}_\infty)$

There is an algebra morphism $\mathcal{U}_q(sl_\infty) \to \mathcal{U}_q(\hat{sl}_\infty)$ defined by $k_h \mapsto k_h$, $x_i^{\pm} \mapsto x_{i,0}^{\pm}$ $(h \in \mathfrak{h}, i \in \mathbb{Z})$. Its image is called the *horizontal quantum affine subalgebra* of $\mathcal{U}_q(\hat{sl}_\infty)$ and is denoted by $\mathcal{U}_q^h(\hat{sl}_\infty)$. In particular, a $\mathcal{U}_q(\hat{sl}_\infty)$-module V has also a structure of a $\mathcal{U}_q(sl_\infty)$-module. We denote by $\mathrm{Res}(V)$ the restricted $\mathcal{U}_q(sl_\infty)$-module obtained from V.

For $J = [a, b] \subseteq \mathbb{Z}$, let us denote by $\mathcal{U}_q(\hat{sl}_\infty)_J$ the subalgebra of $\mathcal{U}_q(\hat{sl}_\infty)$ generated by the $x_{i,r}^{\pm}$, k_h, $h_{i,m}$ $(i \in J,\ r \in \mathbb{Z},\ m \in \mathbb{Z} - \{0\},\ h \in \oplus_{j\in J}\mathbb{Q}h_j)$. It is isomorphic to the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{b-a+2})'$ [Bec94, Dri88].

For $i \in \mathbb{Z}$, the subalgebra $\hat{\mathcal{U}}_i$ generated by the $x_{i,r}^{\pm}, h_{i,m}, k_{ph_i}$ $(r \in \mathbb{Z},\ m \in \mathbb{Z} - \{0\},\ p \in \mathbb{Q})$ is isomorphic to $\mathcal{U}_q(\hat{sl}_2)'$.

We have a triangular decomposition of $\mathcal{U}_q(\hat{sl}_\infty)$.

Theorem 4.2.11. *[Mik00, Nak01] We have an isomorphism of vector spaces*

$$\mathcal{U}_q(\hat{sl}_\infty) \simeq \mathcal{U}_q(\hat{sl}_\infty)^- \otimes \mathcal{U}_q(\hat{\mathfrak{h}}) \otimes \mathcal{U}_q(\hat{sl}_\infty)^+,$$

where $\mathcal{U}_q(\hat{sl}_\infty)^{\pm}$ (resp. $\mathcal{U}_q(\hat{\mathfrak{h}})$) is generated by the $x_{i,r}^{\pm}$ (resp. the k_h, the $h_{i,m}$).

4.2.5 Representations of $\mathcal{U}_q(\hat{sl}_\infty)$

Representations of ℓ-highest weight

Definition 4.2.12. *A representation V of $\mathcal{U}_q(\hat{sl}_\infty)$ is said to be integrable (resp. in the category $\mathcal{O}$) if* $\mathrm{Res}(V)$ *is integrable (resp. in the category $\mathcal{O}$) as a $\mathcal{U}_q(sl_\infty)$-module. Let $\mathcal{O}_{\mathrm{int}}$ be the category of integrable representations of $\mathcal{U}_q(\hat{sl}_\infty)$ in the category $\mathcal{O}$.*

The definition of highest weight representations for $\mathcal{U}_q(\hat{sl}_\infty)$ (and for general quantum affinizations) does not make sense with Drinfeld generators. However an analogous notion of ℓ-highest weight modules exists.

Definition 4.2.13. *A representation V of $\mathcal{U}_q(\hat{sl}_\infty)$ is said to be of ℓ-highest weight if there is $v \in V$ such that*

(i) $V = \mathcal{U}_q(\hat{sl}_\infty)^- \cdot v$,

(ii) there exists $\gamma \in \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}), \mathbb{C})$ *(called ℓ-highest weight) such that*

$$\phi^{\pm}_{i,\pm m} \cdot v = \gamma(\phi^{\pm}_{i,\pm m}) v \text{ for all } i \in \mathbb{Z} \text{ and } m \geq 0,$$

(iii) for any $i \in \mathbb{Z}, r \in \mathbb{Z}$, $x^+_{i,r} \cdot v = 0$.

Such a representation is in the category $\mathcal{O}$ and is infinite-dimensional. For $\gamma \in \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}), \mathbb{C})$, by Theorem 4.2.11 we have a corresponding Verma module $M(\gamma)$ and a simple representation $V(\gamma)$ which are ℓ-highest weight.

Theorem 4.2.14. *[Her11] The simple representations $V(\gamma)$ of $\mathcal{U}_q(\hat{sl}_\infty)$ are integrable if there is $(P_i)_{i\in\mathbb{Z}} \in (1+u\mathbb{C}[u])^{(\mathbb{Z})}$ satisfying for $i \in \mathbb{Z}$ the relation in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$)*

$$\gamma(\phi^{\pm}_i(z)) = q^{deg(P_i)} \frac{P_i(zq^{-1})}{P_i(zq)}.$$

The polynomials P_i are called Drinfeld polynomials and the representation $V(\gamma)$ is then denoted by $V((P_i)_{i\in\mathbb{Z}})$.

The Kirillov-Reshetikhin module associated to $k \geq 0$, $a \in \mathbb{C}^*$ and $\ell \in \mathbb{Z}$, is the simple integrable representation defined by the n–tuple

$$P_i(u) = \begin{cases} (1-ua)(1-uaq^2)\cdots(1-uaq^{2(k-1)}) \text{ for } i = \ell, \\ 1 \text{ for } i \neq \ell. \end{cases}$$

If $k = 1$, it is also called fundamental module.

Theorem 4.2.15. *[Her11] Let v be a ℓ-highest weight vector of the simple $\mathcal{U}_q(\hat{sl}_\infty)$-module $V((P_i)_{i\in\mathbb{Z}})$. Then*

$$V((P_i)_{i\in\mathbb{Z}}) = \bigcup_{n\geq 0} V_n((P_i)_{i\in\mathbb{Z}}) \text{ where } V_n((P_i)_{i\in\mathbb{Z}}) = \mathcal{U}_q(\hat{sl}_\infty)_{[-n,n]} \cdot v.$$

Furthermore each $V_n((P_i)_{i\in\mathbb{Z}})$ is a simple finite-dimensional $\mathcal{U}_q(\hat{sl}_\infty)_{[-n,n]}$-module.

Morphism of q-character

Consider an integrable representation V of $\mathcal{U}_q(\hat{sl}_\infty)$. As the subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ is commutative, the weight spaces V_ν split in simultaneous generalized eigenspaces

$$V_\nu = \bigoplus_{\gamma \in \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}),\mathbb{C})} V_\gamma,$$

where $V_\gamma = \{x \in V / \exists p \in \mathbb{N}, \forall i \in \mathbb{Z}, \forall m \geq 0, (\phi^\pm_{i,\pm m} - \gamma(\phi^\pm_{i,\pm m}))^p \cdot x = 0\}$. If $V_\gamma \neq \{0\}$, γ is called an ℓ-weight of V and V_γ is an ℓ-weight space.

Definition 4.2.16. *A $\mathcal{U}_q(\hat{sl}_\infty)$-module V is weighted if the Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ acts on V by diagonalizable operators. The module V is thin if it is weighted and the joint spectrum is simple.*

Definition 4.2.17. *[FR99, Her05, Nak01] The q–character of an integrable representation V of $\mathcal{U}_q(\hat{sl}_\infty)$ with finite-dimensional ℓ-weight spaces is defined by the formal sum*

$$\chi_q(V) = \sum_{\gamma \in \mathrm{Hom}(\mathcal{U}_q(\hat{\mathfrak{h}}),\mathbb{C})} \dim(V_\gamma) e(\gamma).$$

As in [FJMM13], we set

$$\psi(z) = \frac{q - q^{-1}z}{1-z}.$$

Proposition 4.2.18. *[FR99, Her11] Let V be an integrable representation of $\mathcal{U}_q(\hat{sl}_\infty)$. An ℓ-weight γ of V satisfies the property*

1. *for all $i \in \mathbb{Z}$, there exist $k, l \in \mathbb{N}$ and $a_{1,i}, \cdots, a_{k,i}, b_{1,i}, \cdots, b_{l,i} \in \mathbb{C}^*$ such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$):*

$$\sum_{m \geq 0} \gamma(\phi^\pm_{i,\pm m}) z^{\pm m} = \prod_{1 \leq u \leq k} \psi(a_{u,i} q z) \prod_{1 \leq v \leq l} \psi(b_{v,i} q z)^{-1} = \prod_{1 \leq u \leq k} \psi(a_{u,i} q z) \prod_{1 \leq v \leq l} \psi(b_{v,i}^{-1} q z^{-1})$$

Furthermore if V is in the category $\mathcal{O}$, then

(2) there exist $\omega \in P^+$, $\alpha \in Q^+$ satisfying $\nu = \omega - \alpha$.

Correspondence between ℓ-weights and monomials

Consider formal variables $Y_{i,a}^{\pm 1}$ ($i \in \mathbb{Z}$, $a \in \mathbb{C}^*$) and let A be the group of monomials of the form

$$m = \prod_{i \in \mathbb{Z}, a \in \mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}, \ u_{i,a}(m) \in \mathbb{Z}.$$

Set

$$A_{i,a} = Y_{i,aq^{l-1}} Y_{i,aq^{l+1}} Y^{-1}_{i-1,aq^l} Y^{-1}_{i+1,aq^l} \in A.$$

A monomial m is said to be J-dominant ($J \subset \mathbb{Z}$) if for all $j \in J$ and $a \in \mathbb{C}^*$ we have $u_{j,a}(m) \geq 0$. A $\mathbb{Z}$-dominant monomial is said to be dominant.

Let V be an integrable $\mathcal{U}_q(\hat{sl}_\infty)$-module. For γ an ℓ-weight of V, one defines the monomial $m_\gamma = \prod_{i \in \mathbb{Z}} \prod_{a_i \in \mathbb{C}^*} Y_{i,a_i} \prod_{b_i \in \mathbb{C}^*} Y^{-1}_{i,b_i}$ where

$$\sum_{m \geq 0} \gamma(\phi^\pm_{i,\pm m}) z^{\pm m} = \prod_{a_i} \psi(a_i q z) \prod_{b_i} \psi(b_i q z)^{-1}.$$

We denote $V_\gamma = V_{m_\gamma}$. We rewrite the q–character of an integrable representation V with finite-dimensional ℓ-weight spaces by the formal sum

$$\chi_q(V) = \sum_m \dim(V_m) m \in \mathbb{Z}^A.$$

Let us denote by $\mathcal{M}(V)$ the set of monomials occurring in $\chi_q(V)$.

By this correspondence between ℓ-weights and monomials due to Frenkel-Reshetikhin [FR99], the $\mathbb{Z}$-tuple of Drinfeld polynomials are identified with the dominant monomials. In particular for a dominant monomial m, one denotes by $V(m)$ the simple module of ℓ-highest weight m. For example $V(Y_{\ell,a}Y_{\ell,aq^2} \ldots Y_{\ell,aq^{2(k-1)}})$ is the Kirillov-Reshetikhin module associated to $\ell \in \mathbb{Z}$, $k \geq 0$ and $a \in \mathbb{C}^*$, and $V(Y_{\ell,a})$ is the fundamental module associated to $\ell \in \mathbb{Z}$ and $a \in \mathbb{C}^*$.

Formula of q-characters

In general no formula is known for the q–character of a ℓ-highest weight module of $\mathcal{U}_q(\hat{sl}_\infty)$. But in the special case of Kirillov-Reshetikhin modules, the q–character can be explicitly given as a tableaux sum expression. For that we set for all $a \in \mathbb{C}^*$ and $j \in \mathbb{Z}$

$$\boxed{j}_a = Y^{-1}_{j-1,aq^j} Y_{j,aq^{j-1}}.$$

For $I, J \subset \mathbb{Z}$, a tableaux $(T_{i,j})_{i \in I, j \in J}$ is said to be semi-standard if it has coefficients in $\mathbb{Z}$ which increase relatively to j and strictly increase relatively to i.

Consider $\ell \in \mathbb{Z}$ and $k \geq 0$. Let $\mathcal{T}_{\ell,k}$ be the set of semi-standard tableaux $(T_{i,j})_{i \leq \ell, 1 \leq j \leq k}$ such that for any $1 \leq j \leq k$,

$$T_{i,j} = i \text{ for } i << 0.$$

Theorem 4.2.19. *[Her11] We have the following formula*

$$\chi_q(V(Y_{\ell,a} \cdots Y_{\ell,aq^{2(k-1)}})) = \sum_{T \in \mathcal{T}_{\ell,k}} m_T \tag{4.4}$$

where $m_T = \prod_{i \leq \ell, 1 \leq j \leq k} \boxed{T_{i,j}}_{aq^{\ell-1+2(j-i)}}$.

4.2.6 Extremal loop weight modules

The definition of extremal loop weight modules is given in [Her11] for the quantum toroidal algebras and studied in [Man12c, Man13a]. We introduce its definition in the context of the quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$.

Definition 4.2.20. *An extremal loop weight module of $\mathcal{U}_q(\hat{sl}_\infty)$ of ℓ-weight m is an integrable representation V such that there is $v \in V_m$ satisfying*

(i) $\mathcal{U}_q(\hat{sl}_\infty) \cdot v = V$,

(ii) v is extremal for $\mathcal{U}_q^h(\hat{sl}_\infty)$,

(iii) $\mathcal{U}_q(\hat{sl}_\infty)_J \cdot w$ *is finite-dimensional for all $w \in V$ and $J = [a, b] \subset \mathbb{Z}$ finite.*

Proposition 4.2.21. *Let m be a dominant monomial. Then the simple ℓ-highest weight module $V(m)$ is an extremal loop weight module of* $\mathcal{U}_q(\hat{sl}_\infty)$.

Proof. Let v be a ℓ-highest weight vector of $V(m)$. Denote its weight by $\lambda \in P^+$. As it is recalled above, $V(m) = \mathcal{U}_q(\hat{sl}_\infty) \cdot v$ and is integrable. Furthermore v is a highest weight vector for $\mathcal{U}_q^h(\hat{sl}_\infty)$ which satisfies for all $i \in \mathbb{Z}$,

$$(x_i^-)^{1+\lambda(h_i)} \cdot v = 0.$$

Hence $\mathcal{U}_q^h(\hat{sl}_\infty) \cdot v$ is isomorphic to the simple highest weight $\mathcal{U}_q(sl_\infty)$-module $V(\lambda)$, and v is extremal of weight λ for the horizontal quantum affine subalgebra.

It remains to show that for all $w \in V(m)$ and $J = [a, b] \subset \mathbb{Z}$ finite, $\mathcal{U}_q(\hat{sl}_\infty)_J \cdot w$ is finite-dimensional. But there exists $n \in \mathbb{N}$ such that $J \subset [-n, n]$ and

$$\mathcal{U}_q(\hat{sl}_\infty)_J \cdot w \subset \mathcal{U}_q(\hat{sl}_\infty)_{[-n,n]} \cdot w \subset V_n(m) = \mathcal{U}_q(\hat{sl}_\infty)_{[-n,n]} \cdot v.$$

As $V_n((P_i)_{i\in\mathbb{Z}})$ is finite-dimensional by Theorem 4.2.15, it is also the case of $\mathcal{U}_q(\hat{sl}_\infty)_J \cdot w$. So the $\mathcal{U}_q(\hat{sl}_\infty)$-module $V(m)$ is an extremal loop weight module of ℓ-weight m. □

In this chapter we construct extremal loop weight modules of $\mathcal{U}_q(\hat{sl}_\infty)$ by fusion product of ℓ-highest weight modules and ℓ-lowest weight modules. This process is introduced in Chapter 3 in the quantum toroidal case, and is motivated by the original study of extremal weight modules of quantum Kac-Moody algebras [Kas94]: in fact to study these representations, tensor products of highest weight modules and lowest weight modules are considered (see [Kas94, Lemma 8.2.1]).

4.2.7 Drinfeld coproduct

Let Δ_D be the coproduct defined by ($i \in \mathbb{Z}$)

$$\Delta_D(x_i^+(z)) = x_i^+(z) \otimes 1 + \phi_i^-(z) \otimes x_i^+(z), \tag{4.5}$$

$$\Delta_D(x_i^-(z)) = x_i^-(z) \otimes \phi_i^+(z) + 1 \otimes x_i^-(z), \tag{4.6}$$

$$\Delta_D(\phi_i^\pm(z)) = \phi_i^\pm(z) \otimes \phi_i^\pm(z). \tag{4.7}$$

Note that this map does not define a coproduct in the usual sense because (4.5) and (4.6) involve infinite sums and are not elements of $\mathcal{U}_q(\hat{sl}_\infty) \otimes \mathcal{U}_q(\hat{sl}_\infty)$ (they take values in a completion of it). However it can be used to define a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module on (a subspace of) tensor products of $\mathcal{U}_q(\hat{sl}_\infty)$-modules when the summations are well-defined.

Lemma 4.2.22. *[FJMM13, Man13a] Let V, W be $\mathcal{U}_q(\hat{sl}_\infty)$-modules. Let $U \subseteq V \otimes W$ be a linear subspace such that for any $u \in U$, $\Delta_D(x_i^+(z)) \cdot u$ is well-defined in U ($i \in \mathbb{Z}$). Then the coproduct Δ_D endows U with a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module, called the fusion product of V and W. Furthermore if V and W are integrable then U is an integrable $\mathcal{U}_q(\hat{sl}_\infty)$-module.*

Lemma 4.2.23. *Set $J = [a, b] \subset \mathbb{Z}$ finite, and assume further that for all $v \in V, w \in W$, $\mathcal{U}_q(\hat{sl}_\infty)_J \cdot v$ and $\mathcal{U}_q(\hat{sl}_\infty)_J \cdot w$ are finite-dimensional. Then the $\mathcal{U}_q(\hat{sl}_\infty)$-module U satisfies also*

$$\mathcal{U}_q(\hat{sl}_\infty)_J \cdot u \text{ is finite-dimensional for all } u \in U.$$

The proof is analogue to the one of Lemma 1.8.3.

4.3 Extremal fundamental loop weight modules

In this section, we define a $\mathcal{U}_q(\hat{sl}_\infty)$-module structure on the tensor product of fundamental modules

$$V(Y_{\ell,a}) \otimes V(Y_{0,aq^\ell}^{-1}) \text{ with } \ell \geq 1 \text{ and } a \in \mathbb{C}^*$$

by using the Drinfeld coproduct Δ_D. We obtain in that way new integrable modules $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ with basis labelled by semi-standard tableaux of shape (ℓ). We show in particular that these modules are ℓ-extremal. We call them the extremal fundamental loop weight modules of $\mathcal{U}_q(\hat{sl}_\infty)$.

The fusion product process used here is inspired to the one in Chapter 3 for the toroidal case. As it is said above, the representation theory of $\mathcal{U}_q(\hat{sl}_\infty)$ and of quantum toroidal algebras are very different: the fundamental $\mathcal{U}_q(\hat{sl}_\infty)$-modules are thin, which is not the case for the quantum toroidal algebras $\mathcal{U}_q(sl_{n+1}^{tor})$ (see [Her11, Section 4.1]). Let us point out that these differences have consequences in our work: we obtain here more precise results for the fusion product of $\mathcal{U}_q(\hat{sl}_\infty)$-modules. Another difficulty met in Chapter 3 is that the cyclicity of the Dynkin diagram associated to $\mathcal{U}_q(sl_{n+1}^{tor})$ bring some restricted conditions in the construction of extremal loop weight modules (see Proposition 3.2.11). We will see that it is possible here to associate extremal fundamental loop weight $\mathcal{U}_q(\hat{sl}_\infty)$-modules for all the nodes of the Dynkin diagram of type A_∞.

In the first subsection, we give explicit formulae of the action on the fundamental $\mathcal{U}_q(\hat{sl}_\infty)$-modules, using the $\mathcal{U}_q(sl_\infty)$-crystal structure on the set of semi-standard tableaux.

In the second subsection, we construct the fusion product of the $\mathcal{U}_q(\hat{sl}_\infty)$-modules $V(Y_{\ell,a})$ and $V(Y_{0,aq^\ell}^{-1})$, the action of $\mathcal{U}_q(\hat{sl}_\infty)$ being defined via the Drinfeld coproduct (Proposition 4.3.2). We define a submodule $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ of it, which we call the extremal fundamental loop weight module.

We study the $\mathcal{U}_q(\hat{sl}_\infty)$-modules $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ in the third subsection: these representations are integrable with basis labelled by semi-standard tableaux of shape (ℓ). The action of $\mathcal{U}_q(\hat{sl}_\infty)$ on it is described by the $\mathcal{U}_q(sl_\infty)$-crystal structure on the set of semi-standard tableaux of shape (ℓ) (Theorem 4.3.10). Furthermore $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is an extremal loop weight module of ℓ-weight $Y_{\ell,a}Y_{0,aq^\ell}^{-1}$ (Theorem 4.3.14) which is irreducible as a $\mathcal{U}_q(\hat{sl}_\infty)$-module and as a $\mathcal{U}_q^h(\hat{sl}_\infty)$-module (Proposition 4.3.12).

4.3.1 Action on the fundamental $\mathcal{U}_q(\hat{sl}_\infty)$-modules

In this section we give explicit formulae of the action on the fundamental $\mathcal{U}_q(\hat{sl}_\infty)$-modules $V(Y_{\ell,a})$ ($\ell \in \mathbb{Z}, a \in \mathbb{C}^*$).

Let $\mathcal{T}_\ell^+ = \mathcal{T}_{\ell,1}$ be the set of semi-standard tableaux

$$T = (\cdots < i_{\ell-2} < i_{\ell-1} < i_\ell) \text{ such that } i_j = j \text{ for } j << 0.$$

We endow $\mathcal{T}_\ell^+$ with a structure of $\mathcal{U}_q(sl_\infty)$-crystal: the weight function is defined by setting

$$\mathrm{wt}(T) = \sum_{j \leq \ell} \Lambda_{i_j} - \Lambda_{i_j - 1}$$

for all $T = (\cdots < i_{\ell-2} < i_{\ell-1} < i_\ell) \in \mathcal{T}_\ell^+$. The Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ are described in terms of tableaux: for $i \in \mathbb{Z}$ we have $\tilde{e}_i \cdot T = T'$ or 0. Here T' is obtained from T by replacing $i+1$ by i. If it is not possible (i.e. when we have both $i+1$ and i in T or when $i+1$ does not occur in T), then it is zero. Similarly $\tilde{f}_i \cdot m_{T;j} = m_{T'';j}$ or 0, where T'' is given by replacing i by $i+1$.

Proposition 4.3.1. *Let $\ell \in \mathbb{Z}$ and $a \in \mathbb{C}^*$ and consider the fundamental $\mathcal{U}_q(\hat{sl}_\infty)$-module $V(Y_{\ell,a})$. Then there exists a basis of $V(Y_{\ell,a})$ indexed by $\mathcal{T}_\ell^+$ such that the action of $\mathcal{U}_q(\hat{sl}_\infty)$ is given by (for convenience in the calculations, we remind the complex number $a \in \mathbb{C}^*$ in subscript)*

$$\begin{aligned}
x_i^+(z) \cdot T_a &= \sum_p \delta_{\{i_p=i+1\}} \delta(aq^{i+\ell+1-2p}z)(\tilde{e}_i \cdot T)_a, \\
x_i^-(z) \cdot T_a &= \sum_p \delta_{\{i_p=i\}} \delta(aq^{i+\ell+1-2p}z)(\tilde{f}_i \cdot T)_a, \\
\phi_i^\pm(z) \cdot T_a &= \begin{cases} \psi(aq^{i+\ell+3-2p}z)^{-1} \cdot T_a & \begin{array}{c}\text{if there exists } p \le \ell \text{ such that} \\ i_p = i+1 \text{ and } i_{p-1} \neq i,\end{array} \\ \psi(aq^{i+\ell+1-2p}z) \cdot T_a & \begin{array}{c}\text{if there exists } p \le \ell \text{ such that} \\ \text{if } i_p = i \text{ and } i_{p+1} \neq i+1,\end{array} \\ T_a & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.8}$$

where $T = (\cdots < i_{\ell-2} < i_{\ell-1} < i_\ell)$ is a semi-standard tableaux in $\mathcal{T}_\ell^+$.

In particular for all $T \in \mathcal{T}_\ell^+$, T_a is an ℓ-weight vector of weight m_T.

Proof. This result is known for the fundamental $\mathcal{U}_q(\hat{sl}_{n+1})$-modules (see Theorem 2.3.12). We obtain the result for $\mathcal{U}_q(\hat{sl}_\infty)$ by inductive limit, using Theorem 4.2.15. ☐

We have an analogue result for the ℓ-lowest weight modules. For that let $\mathcal{T}_s^-$ $(s \in \mathbb{Z})$ be the set of semi-standard tableaux

$$T = (i_{s+1} < i_{s+2} < i_{s+3} < \cdots) \text{ such that } i_j = j \text{ for } j >> 0.$$

Then $(\mathcal{T}_s^-, \mathrm{wt}, \tilde{e}_i, \tilde{f}_i)$ is a $\mathcal{U}_q(sl_\infty)$-crystal. And for all $a \in \mathbb{C}^*$, there exists a basis of $V(Y_{s,a}^{-1})$ indexed by $\mathcal{T}_s^-$ such that the action of $\mathcal{U}_q(\hat{sl}_\infty)$ on it is given for all $T = (i_{s+1} < i_{s+2} < i_{s+3} < \cdots) \in \mathcal{T}_s^-$ by

$$\begin{aligned}
x_i^+(z) \cdot T_a &= \sum_p \delta_{\{i_p=i+1\}} \delta(aq^{i+s+1-2p}z)(\tilde{e}_i \cdot T)_a, \\
x_i^-(z) \cdot T_a &= \sum_p \delta_{\{i_p=i\}} \delta(aq^{i+s+1-2p}z)(\tilde{f}_i \cdot T)_a, \\
\phi_i^\pm(z) \cdot T_a &= \begin{cases} \psi(aq^{i+s+3-2p}z)^{-1} \cdot T_a & \begin{array}{c}\text{if there exists } p \ge s+1 \text{ such that} \\ i_p = i+1 \text{ and } i_{p-1} \neq i,\end{array} \\ \psi(aq^{i+s+1-2p}z) \cdot T_a & \begin{array}{c}\text{if there exists } p \ge s+1 \text{ such that} \\ \text{if } i_p = i \text{ and } i_{p+1} \neq i+1,\end{array} \\ T_a & \text{otherwise.} \end{cases}
\end{aligned}$$

4.3.2 Fusion product of fundamental modules

Proposition 4.3.2. *Let $\ell \geq 1$ and $a \in \mathbb{C}^*$. The coproduct Δ_D is well-defined on the tensor product $V(Y_{\ell,a}) \otimes V(Y_{0,aq^\ell}^{-1})$ and it endows it with a structure of integrable $\mathcal{U}_q(\hat{sl}_\infty)$-module.*

Remark 4.3.3. *We have introduced a fusion product process in Section 3.2 to define new integrable modules for the quantum toroidal algebras. This process can also be used in our situation to show Proposition 4.3.2. We give here a direct proof of it for the convenience of the reader. As it is pointed out above, the computations are simpler in the case of the quantum affinization $\mathcal{U}_q(\hat{sl}_\infty)$, stemming from the thin property of the fundamental $\mathcal{U}_q(\hat{sl}_\infty)$-modules.*

Proof. Let us consider

$$T = (\cdots < i_{\ell-2} < i_{\ell-1} < i_\ell) \in \mathcal{T}_\ell^+ \text{ and } T' = (j_1 < j_2 < j_3 < \cdots) \in \mathcal{T}_0^-.$$

We study the action of $x_i^+(z)$ on $T_a \otimes T'_{aq^\ell} \in V(Y_{\ell,a}) \otimes V(Y_{0,aq^\ell}^{-1})$:

$$x_i^+(z) \cdot (T_a \otimes T'_{aq^\ell}) = \textstyle\sum_r \delta_{\{i_r=i+1\}}\delta(aq^{i+\ell+1-2r}z)(\tilde{e}_i \cdot T)_a \otimes T'_{aq^\ell} +$$
$$\sum_{r,s}\Big[1 + \delta_{\{i_r=i+1\}}\delta_{\{i_{r-1}\neq i\}}\Big(\psi(q^{2(s-r)+2})^{-1} - 1\Big) + \delta_{\{i_r=i\}}\delta_{\{i_{r+1}\neq i+1\}}\Big(\psi(q^{2(s-r)}) - 1\Big)\Big]$$
$$\times \delta_{\{j_s=i+1\}}\delta(aq^{i+\ell+1-2r}z)T_a \otimes (\tilde{e}_i \cdot T')_{aq^\ell}.$$

By definition of $\mathcal{T}_\ell^+$ and $\mathcal{T}_0^-$, we have $r \leq i$, $s \geq i+1$, and $s-r \geq 1$. Hence the coefficients of the series $x_i^+(z) \cdot (T_a \otimes T'_{aq^\ell})$ are well-defined. The result follows by Lemma 4.2.22. $\square$

Let us fix $\ell \in \mathbb{Z}$. For $T = (\cdots < i_{\ell-2} < i_{\ell-1} < i_\ell)$ a semi-standard tableaux in $\mathcal{T}_\ell^+$, we define

$$\alpha_T = \sup\{p \leq \ell | i_p = p\}.$$

Note that α_T is well-defined by definition of $\mathcal{T}_\ell^+$. In the same way, we define for all $T = (i_1 < i_2 < i_3 < \cdots) \in \mathcal{T}_0^-$

$$\beta_T = \inf\{p \geq 1 | i_p = p\}.$$

Assume that $\ell \geq 1$. We have endowed the sets $\mathcal{T}_\ell^+$ and $\mathcal{T}_0^-$ with a structure of $\mathcal{U}_q(sl_\infty)$-crystal. By the tensor product rules (see [Kas94, Kas02a]), $\mathcal{T}_\ell^+ \otimes \mathcal{T}_0^-$ is a $\mathcal{U}_q(sl_\infty)$-crystal. Let us denote by $\mathcal{T}_\ell$ the subset of $\mathcal{T}_\ell^+ \otimes \mathcal{T}_0^-$ consisting of

$$T \otimes T' \text{ with } T \in \mathcal{T}_\ell^+, T' \in \mathcal{T}_0^- \text{ and } \alpha_T \geq \beta_{T'} - 1.$$

Proposition 4.3.4. *The set $\mathcal{T}_\ell$ is the connected sub-$\mathcal{U}_q(sl_\infty)$-crystal of $\mathcal{T}_\ell^+ \otimes \mathcal{T}_0^-$ generated by $(\cdots < \ell-2 < \ell-1 < \ell) \otimes (1 < 2 < 3 < \cdots)$.*

Proof. Set $T \otimes T' \in \mathcal{T}_\ell$ and $i \in \mathbb{Z}$. Let us show that $\tilde{e}_i \cdot (T \otimes T') \in \mathcal{T}_\ell \cup \{0\}$ (the proof for the Kashiwara operators $\tilde{f}_i$ is analogous). We have the following equalities

$$\alpha_{\tilde{e}_i \cdot T} = \begin{cases} \alpha_T & \text{if } \tilde{e}_i \cdot T' \neq 0 \text{ and } i > \alpha_T + 1, \\ \alpha_T + 1 & \text{if } i = \alpha_T + 1, \end{cases}$$
$$\beta_{\tilde{e}_i \cdot T'} = \begin{cases} \beta_{T'} & \text{if } \tilde{e}_i \cdot T' \neq 0 \text{ and } i < \beta_{T'} - 1, \\ \beta_{T'} + 1 & \text{if } i = \beta_{T'} - 1. \end{cases}$$

In particular $\alpha_{\tilde{e}_i \cdot T} \leq \alpha_T$. So $\tilde{e}_i \cdot (T \otimes T')$ belongs to $\mathcal{T}_\ell \cup \{0\}$, except perhaps when $\alpha_T = \beta_{T'} - 1 = i$. But in that case, we have by the tensor product rules

$$\tilde{e}_i \cdot (T \otimes T') = (\tilde{e}_i \cdot T) \otimes T' = 0 \in \mathcal{T}_\ell \cup \{0\}.$$

Hence $\mathcal{T}_\ell$ is a sub-$\mathcal{U}_q(sl_\infty)$-crystal of $\mathcal{T}_\ell^+ \otimes \mathcal{T}_0^-$. And one can show by straightforward computations that it is the connected sub-$\mathcal{U}_q(sl_\infty)$-crystal of $\mathcal{T}_\ell^+ \otimes \mathcal{T}_0^-$ generated by

$$(\cdots < \ell-2 < \ell-1 < \ell) \otimes (1 < 2 < 3 < \cdots).$$

$\square$

Denote by $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ ($\ell \geq 1, a \in \mathbb{C}^*$) the subvector space of $V(Y_{\ell,a}) \otimes V(Y_{0,aq^\ell}^{-1})$ generated by $T_a \otimes T'_{aq^\ell}$ with $T \otimes T' \in \mathcal{T}_\ell$.

Proposition 4.3.5. *The coproduct Δ_D endows $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ with a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module. We call it the extremal fundamental loop weight module*[1].

Proof. The action of $x_i^\pm(z)$ ($i \in \mathbb{Z}$) is well-defined on $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ by Proposition 4.3.2. By Lemma 4.2.22, it remains to show that $x_i^+(z)\cdot(T_a \otimes T'_{aq^\ell}) \in V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ for all $T \otimes T' \in \mathcal{T}_\ell$ (the proof being similar for $x_i^-(z)$). As in the last proof, it holds except perhaps when $\alpha_T = \beta_{T'} - 1 = i$. In that case we have

$$x_i^+(z) \cdot T_a \otimes T'_{aq^\ell} = \psi(q^2)\delta(aq^{\ell-i-1}z) \cdot T_a \otimes (\tilde{e}_i \cdot T')_{aq^\ell} = 0 \in V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$$

Hence the coproduct Δ_D endows $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ with a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module. $\square$

4.3.3 Study of the extremal fundamental loop weight modules

Let $\mathcal{T}_{[1,\ell]}$ be the set of semi-standard tableaux $T = (i_1 < i_2 < \cdots < i_\ell)$ of shape (ℓ). We consider the application $\Pi : \mathcal{T}_\ell \to \mathcal{T}_{[1,\ell]}$ defined for all $T \otimes T' \in \mathcal{T}_\ell$ by

$$\Pi(T \otimes T') = (j_1 < j_2 < \cdots < j_{\alpha_T} < i_{\alpha_T+1} < \cdots < i_\ell)$$

where $T = (\cdots < i_{\ell-2} < i_{\ell-1} < i_\ell)$ and $T' = (j_1 < j_2 < j_3 < \cdots)$. This application is bijective, its inverse sending $T = (i_1 < i_2 < \cdots < i_\ell) \in \mathcal{T}_{[1,\ell]}$ on

$$(\cdots < 0 < \max(i_1,1) < \cdots < \max(i_\ell,\ell)) \otimes (\min(i_1,1) < \cdots < \min(i_\ell,\ell) < \ell+1 < \cdots).$$

Proposition 4.3.6. *The application $\Pi : \mathcal{T}_\ell \to \mathcal{T}_{[1,\ell]}$ is an isomorphism of $\mathcal{U}_q(sl_\infty)$-crystals.*

Proof. Let $T = (\cdots < i_{\ell-2} < i_{\ell-1} < i_\ell)$ and $T' = (j_1 < j_2 < j_3 < \cdots)$ be semi-standard tableaux such that $T \otimes T' \in \mathcal{T}_\ell$, and $i \in \mathbb{Z}$. We have to consider the following cases

- $i > \alpha_T$: then we have $\beta_{T'} \leq i$ and $i, i+1 \in T'$. By the tensor product rules, we have

$$\tilde{f}_i \cdot (T \otimes T') = (\tilde{f}_i \cdot T) \otimes T'.$$

Hence,

$$\begin{aligned}\Pi(\tilde{f}_i \cdot (T \otimes T')) &= \begin{cases} (j_1 < \cdots < i_p + 1 < \cdots < i_\ell) & \text{if there exists } p > \alpha_T \text{ such that } i_p = i \text{ and } i_{p+1} \neq i+1, \\ 0 \text{ otherwise} \end{cases} \\ &= \tilde{f}_i \cdot \Pi(T \otimes T')\end{aligned}$$

1. We will pay attention that the modules $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ are defined here over $\mathcal{U}_q(\hat{sl}_\infty)$, and not over the quantum toroidal algebras (Chapter 2 and Chapter 3).

- $i < \alpha_T$: then we have $i, i+1 \in T$ and by the tensor product rules,
$$\tilde{f}_i \cdot (T \otimes T') = T \otimes (\tilde{f}_i \cdot T').$$
Furthermore, $j_{\alpha_T} \leq j_{\alpha_T+1} - 1 = \alpha_T$. Hence,
$$\begin{aligned}\Pi(\tilde{f}_i \cdot (T \otimes T')) &= \begin{cases}(j_1 < \cdots < j_p + 1 < \cdots < i_\ell) & \begin{array}{c}\text{if there exists } 1 \leq p < \alpha_T \text{ such that}\\ i_p = i \text{ and } i_{p+1} \neq i+1,\end{array}\\ 0 \text{ otherwise}\end{cases}\\ &= \tilde{f}_i \cdot \Pi(T \otimes T')\end{aligned}$$
- $i = \alpha_T$: then we have $i+1 \notin T$ and $\beta_{T'} \leq i+1$.
Assume that $\beta_{T'} < i+1$. In that case, $i, i+1 \in T'$ and we have
$$\tilde{f}_i \cdot (T \otimes T') = (\tilde{f}_i \cdot T) \otimes T'.$$
Furthermore $\alpha_{\tilde{f}_i \cdot T} = \alpha_T - 1$ and $j_{\alpha_T} = i$. So we have
$$\begin{aligned}\Pi(\tilde{f}_i \cdot (T \otimes T')) &= (j_1 < \cdots < j_{\alpha_T - 1} < i+1 < i_{\alpha_T+1} < \cdots < i_\ell)\\ &= \tilde{f}_i \cdot (j_1 < \cdots < j_{\alpha_T-1} < i = j_{\alpha_T} < i_{\alpha_T+1} < \cdots < i_\ell)\\ &= \tilde{f}_i \cdot \Pi(T \otimes T').\end{aligned}$$
If $\beta_{T'} = i+1$, we have $i \notin T'$. By the tensor product rules we get
$$\tilde{f}_i \cdot (T \otimes T') = T \otimes (\tilde{f}_i \cdot T') = 0.$$
Furthermore we have $\alpha_T = \beta_{T'} - 1$ and $j_{\alpha_T} < \alpha_T = i < i_{\alpha_T+1}$. Then $i \notin \Pi(T \otimes T')$ and $\tilde{f}_i \cdot \Pi(T \otimes T') = 0$.

As the application $\Pi : \mathcal{T}_\ell \to \mathcal{T}_{[1,\ell]}$ is also bijective, this is an isomorphism of $\mathcal{U}_q(sl_\infty)$-crystals. □

Remark 4.3.7. *Let us describe this isomorphism at the level of the monomial crystals (see [Kas03, Man12c, Nak03] for its definition). For that denote by $\mathcal{M}_\ell$ the sub-$\mathcal{U}_q(sl_\infty)$-crystal of $\mathcal{M}(Y_{\ell,a}) \otimes \mathcal{M}(Y_{0,aq^\ell}^{-1})$ generated by $Y_{\ell,a} \otimes Y_{0,aq^\ell}^{-1}$. Let us consider also $\mathcal{M}(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ the monomial crystal generated by $Y_{\ell,a}Y_{0,aq^\ell}^{-1}$. By that we have done above, we have*
$$\mathcal{M}(Y_{\ell,a}Y_{0,aq^\ell}^{-1}) = \{m_T = \prod_{1 \leq j \leq \ell} \boxed{i_j}_{aq^{\ell+1-2j}} \mid T = (i_1 < i_2 < \cdots < i_\ell) \in \mathcal{T}_{[1,\ell]}\}$$
and
$$\mathcal{M}_\ell = \{m_T \otimes m_{T'} | T \otimes T' \in \mathcal{T}_\ell\}.$$
In the monomial realization, Π is simply given by the product in A
$$\Pi(m_1 \otimes m_2) = m_1 \times m_2 \text{ with } m_1 \otimes m_2 \in \mathcal{M}_\ell.$$

As a direct consequence, we have

Proposition 4.3.8. *Let $\ell \geq 1$ and $a \in \mathbb{C}^*$. The $\mathcal{U}_q(\hat{sl}_\infty)$-module $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is thin, and we have*
$$\chi_q(V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})) = \sum_{T \in \mathcal{T}_{[1,\ell]}} m_T \tag{4.9}$$
where $m_T = \prod_{1 \leq j \leq \ell} \boxed{i_j}_{aq^{\ell+1-2j}}$.

Proof. Let us recall that we have a basis of ℓ-weight vectors of $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$: it is given by the $T_a \otimes T'_{aq^\ell}$ in $\mathcal{T}_\ell$. Furthermore by (4.7) and the remark above, the ℓ-weight of the vector $T_a \otimes T'_{aq^\ell}$ is

$$m_T \times m_{T'} = m_{\Pi(T\otimes T')}.$$

Hence we have

$$\chi_q(V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})) = \sum_{T\in\mathcal{T}_{[1,\ell]}} m_T.$$

□

Remark 4.3.9. *In particular we have shown here that the monomial crystal $\mathcal{M}(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ generated by $Y_{\ell,a}Y_{0,aq^\ell}^{-1}$ is closed in the sense of Definition 2.2.6.*

By the bijection Π, we parametrize the basis of $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ by the $\mathcal{U}_q(sl_\infty)$-crystal $\mathcal{T}_{[1,\ell]}$. Let us explicit formulae of the action on it.

Theorem 4.3.10. *The action of $\mathcal{U}_q(\hat{sl}_\infty)$ on $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is given for all $T=(i_1<i_2<\cdots<i_\ell)\in\mathcal{T}_{[1,\ell]}$ by*

$$\begin{aligned}
x_i^+(z)\cdot T_a &= \sum_{1\le p\le\ell} \delta_{\{i_p=i+1\}}\delta(aq^{i+\ell+1-2p}z)(\tilde{e}_i\cdot T)_a,\\
x_i^-(z)\cdot T_a &= \sum_{1\le p\le\ell} \delta_{\{i_p=i\}}\delta(aq^{i+\ell+1-2p}z)(\tilde{f}_i\cdot T)_a,\\
\phi_i^\pm(z)\cdot T_a &= \begin{cases} \psi(aq^{i+\ell+3-2p}z)^{-1}\cdot T_a & \text{if there exists } 1\le p\le\ell \text{ such that } i_p=i+1 \text{ and } i_{p-1}\neq i,\\ \psi(aq^{i+\ell+1-2p}z)\cdot T_a & \text{if there exists } 1\le p\le\ell \text{ such that if } i_p=i \text{ and } i_{p+1}\neq i+1,\\ T_a & \text{otherwise.}\end{cases}
\end{aligned} \tag{4.10}$$

Remark 4.3.11. *Note that these formulae are identical to the one (4.8) obtained for the fundamental $\mathcal{U}_q(\hat{sl}_\infty)$-modules. Actually this result generalizes Theorem 2.3.12 for the extremal fundamental loop weight modules of $\mathcal{U}_q(sl_{n+1}^{tor})$. In fact one can rewrite the action on $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ in the context of monomial realizations: it is completely describe by the monomial $\mathcal{U}_q(sl_\infty)$-crystal $\mathcal{M}(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ and the formulae given in Theorem 2.3.12.*

Proof. We have shown that the $\mathcal{U}_q(\hat{sl}_\infty)$-module $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is thin. So to determine the action on this module, it suffices to know the action of the horizontal quantum affine subalgebra $\mathcal{U}_q^h(\hat{sl}_\infty)$. The action of all the algebra $\mathcal{U}_q(\hat{sl}_\infty)$ can be deduced from it by (4.2) and (4.3).

So let us determine the action of $x_{i,0}^+$ ($i\in\mathbb{Z}$) on $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ (we proceed in the same way for $x_{i,0}^-$). Let $T=(\cdots<i_{\ell-2}<i_{\ell-1}<i_\ell)$ and $T'=(j_1<j_2<j_3<\cdots)$ be semi-standard tableaux such that $T\otimes T'\in\mathcal{T}_\ell$. Denote by $(T\otimes T')_a$ the vector $T_a\otimes T'_{aq^\ell}\in V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$. Then $x_{i,0}^-\cdot(T\otimes T')_a$ is equal to

$$\sum_{s\ge1}\delta_{\{j_s=i\}}\cdot T_a\otimes(\tilde{f}_i\cdot T')_{aq^\ell} + (\tilde{f}_i\cdot T)_a\otimes T'_{aq^\ell}\cdot\sum_{r\le\ell,s\ge1}\delta_{\{i_r=i\}}\times$$
$$\left(1+\delta_{\{j_s=i+1\}}\delta_{\{j_{s-1}\neq i\}}\left(\psi(q^{2(r-s)+2})^{-1}-1\right)+\delta_{\{j_s=i\}}\delta_{\{j_{s+1}\neq i+1\}}\left(\psi(q^{2(r-s)})-1\right)\right).$$

We have to consider the following cases

- $i > \alpha_T$: then $i, i+1 \in T'$. So we get

$$x_{i,0}^- \cdot (T \otimes T')_a = (\tilde{f}_i \cdot T)_a \otimes T'_{aq^\ell} = \left(\tilde{f}_i \cdot (T \otimes T')\right)_a.$$

- $i < \alpha_T$: then we have $i, i+1 \in T$ and

$$x_{i,0}^- \cdot (T \otimes T')_a = T_a \otimes (\tilde{f}_i \cdot T')_{aq^\ell} = \left(\tilde{f}_i \cdot (T \otimes T')\right)_a.$$

- $i = \alpha_T$: then we have $i \in T, i+1 \notin T$ and $\beta_{T'} \leq i+1$.
  Assume that $\beta_{T'} < i+1$. In this case, $i, i+1 \in T'$ and we get

$$x_{i,0}^- \cdot (T \otimes T')_a = (\tilde{f}_i \cdot T)_a \otimes T'_{aq^\ell} = \left(\tilde{f}_i \cdot (T \otimes T')\right)_a.$$

  If $\beta_{T'} = i+1$, we have $i \notin T', \alpha_T = \beta_{T'} - 1$ and

$$\begin{aligned} x_{i,0}^- \cdot (T \otimes T')_a &= \psi(q^{2(\alpha_T - \beta_{T'})+2})^{-1}(\tilde{f}_i \cdot T)_a \otimes T'_{aq^\ell} = \psi(1)^{-1}(\tilde{f}_i \cdot T)_a \otimes T'_{aq^\ell} = 0 \\ &= \left(\tilde{f}_i \cdot (T \otimes T')\right)_a. \end{aligned}$$

So we have shown that $x_{i,0}^- \cdot (T \otimes T')_a = \left(\tilde{f}_i \cdot (T \otimes T')\right)_a$. We proceed in the same way for $x_{i,0}^+$. □

As consequences of this theorem, we obtain

Proposition 4.3.12. *Fix $\ell \geq 1$ and $a \in \mathbb{C}^*$. Then $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is irreducible as a $\mathcal{U}_q(\hat{sl}_\infty)$-module and as a $\mathcal{U}_q^h(\hat{sl}_\infty)$-module.*

To prove this proposition, we use the following result (which is an analogue of Lemma 2.3.8).

Lemma 4.3.13. *Let $\mathcal{B}$ be a $\mathcal{U}_q(sl_\infty)$-crystal. Assume that V is a $\mathcal{U}_q(sl_\infty)$-module with basis $(v_b)_{b\in\mathcal{B}}$ indexed by $\mathcal{B}$ which satisfies*

$$\mathrm{wt}(v_b) = \mathrm{wt}(b),\ (x_i^+)^{(k)} \cdot v_b = v_{\tilde{e}_i^k \cdot b} \text{ and } (x_i^-)^{(k)} \cdot v_b = v_{\tilde{f}_i^k \cdot b} \tag{4.11}$$

for all $b \in \mathcal{B}, i \in I$ and $k \in \mathbb{N}$, where $v_0 = 0$ by convention. If the element $b \in \mathcal{B}$ is extremal of weight λ, then the vector v_b is an extremal vector of weight λ. Furthermore if the crystal $\mathcal{B}$ is connected, then the $\mathcal{U}_q(sl_\infty)$-module V is cyclic generated by any v_b with $b \in \mathcal{B}$.

The proof is similar to the one of Lemma 2.3.8. We recall it for the convenience of the reader.

Proof. Assume that $b \in \mathcal{B}$ is extremal of weight λ: there exists $\{b_w\}_{w\in W}$ such that $b_{Id} = b$ and

$$\begin{aligned} \tilde{e}_i \cdot b_w = 0 \text{ and } (\tilde{f}_i)^{w(\lambda)(h_i)} \cdot b_w = b_{s_i(w)} \text{ if } w(\lambda)(h_i) \geq 0, \\ \tilde{f}_i \cdot b_w = 0 \text{ and } (\tilde{e}_i)^{-w(\lambda)(h_i)} \cdot b_w = b_{s_i(w)} \text{ if } w(\lambda)(h_i) \leq 0. \end{aligned} \tag{4.12}$$

For all $w \in W$, set $v_w = v_{b_w}$. By (4.11) and (4.12), $\{v_w\}_{w\in W}$ satisfies $v_{Id} = v_m$ and

$$x_i^\pm \cdot v_w = 0 \text{ if } \pm w(\lambda)(h_i) \geq 0 \text{ and } (x_i^\mp)^{(\pm w(\lambda)(h_i))} \cdot v_w = v_{s_i(w)}.$$

Hence the vector v_m is extremal of weight λ.

Assume that the crystal $\mathcal{B}$ is connected and fix $b \in \mathcal{B}$. For $b' \in \mathcal{B}$, there exists a product s of Kashiwara operators such that $s(b) = b'$. Consider the corresponding operator $S \in \mathcal{U}_q(sl_\infty)$ at the level of V, i.e. S has the same expression as s where the operators $\tilde{e}_i^k$ (resp. $\tilde{f}_i^k$) are replaced by $(x_i^+)^{(k)}$ (resp. $(x_i^-)^{(k)}$) in the product ($k \in \mathbb{N}, i \in I$). By (4.11), $S(v_b) = v_{s(b)} = v_{b'}$ and the $\mathcal{U}_q(sl_\infty)$-module V is cyclic generated by v_b. □

We are now able to prove Proposition 4.3.12.

Proof. Let us consider a non trivial sub-$\mathcal{U}_q^h(\hat{sl}_\infty)$-module V of $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$. By the q–character formula (4.9) all the weight spaces of $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ are of dimension one. So there exists $T \in \mathcal{T}_{[1,\ell]}$ such that $T_a \in V$. By Lemma 4.3.13, the vector T_a generates $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ and $V = V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$. Hence $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is simple as a $\mathcal{U}_q(\hat{sl}_\infty)$-module and as a $\mathcal{U}_q^h(\hat{sl}_\infty)$-module. □

Theorem 4.3.14. *Fix $\ell \geq 1$ and $a \in \mathbb{C}^*$. Then $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is an extremal loop weight $\mathcal{U}_q(\hat{sl}_\infty)$-module generated by the vector $(1 < 2 < \cdots < \ell)_a$ of ℓ-weight $Y_{\ell,a}Y_{0,aq^\ell}^{-1}$.*

Proof. By the fusion product construction, $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is an integrable $\mathcal{U}_q(\hat{sl}_\infty)$-module. As it is also irreducible, $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ is generated by the vector $T = (1 < 2 < \cdots < \ell)_a$. The extremality of this vector for the horizontal quantum affine subalgebra follows from Lemma 4.3.13 and the fact that T is an extremal element in $\mathcal{T}_{[1,\ell]}$. So it remains to prove that for all $w \in V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ and $J = [a,b] \subset \mathbb{Z}$ finite, $\mathcal{U}_q(\hat{sl}_\infty)_J \cdot w$ is finite-dimensional: this is a direct consequence of Lemma 4.2.23. □

Remark 4.3.15. *Set $n \in \mathbb{N}^*, \ell \geq 1$ and $a \in \mathbb{C}^*$. Let us consider the $\mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]}$-modules*

$$V(Y_{\ell,a})_{[-n,\ell+n]} = \mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]} \cdot v^+ \text{ and } V(Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]} = \mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]} \cdot v^-$$

where v^+ and v^- are ℓ-highest weight vector and ℓ-lowest weight vector of the $\mathcal{U}_q(\hat{sl}_\infty)$-modules $V(Y_{\ell,a})$ and $V(Y_{0,aq^\ell}^{-1})$ respectively. Using the result of this section, we deduce that Δ_D endows the tensor product

$$V(Y_{\ell,a})_{[-n,\ell+n]} \otimes V(Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]}$$

with a structure of $\mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]}$-module. Let us set

$$V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]} = \mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]} \cdot v^+ \otimes v^- \subset V(Y_{\ell,a})_{[-n,\ell+n]} \otimes V(Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]}.$$

We give some facts about the $\mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]}$-module $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]}$ (consequences of the study done in this section): it is irreducible and finite-dimensional. In particular, this is a simple ℓ-highest weight $\mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]}$-module. Its q–character is

$$\chi_q(V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]}) = \sum_{T \in (\mathcal{T}_\ell)_{[-n,\ell+n]}} m_T$$

where $(\mathcal{T}_\ell)_{[-n,\ell+n]}$ is the finite set of semi-standard tableaux

$$T = (-n \leq i_1 < i_2 < \cdots < i_\ell \leq \ell + n + 1).$$

The ℓ-highest weight of $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]}$ corresponds to the dominant monomial in its q–character formula. It is obtained for $T = (-n < -n+1 < \cdots < -n+\ell-1)$. Hence we have

$$V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]} = V(m_T)_{[-n,\ell+n]} = V(Y_{-n+\ell-1,aq^{-n-1}})_{[-n,\ell+n]}.$$

Then $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})_{[-n,\ell+n]}$ is the fundamental $\mathcal{U}_q(\hat{sl}_\infty)_{[-n,\ell+n]}$-module of ℓ-highest weight $Y_{-n+\ell-1,aq^{-n-1}}$.

4.4 Tensor product of extremal fundamental loop weight modules

In this section we study tensor products of extremal fundamental loop weight modules $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell})$ with $\ell \geq 1$ and $a \in \mathbb{C}^*$. We obtain new families of extremal loop weight modules with basis labelled by semi-standard tableaux.

In the first subsection, we determine existence conditions of $\mathcal{U}_q(\hat{sl}_\infty)$-module structure on the tensor product of extremal fundamental loop weight modules $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell})$.

In the second subsection we study the case of fusion products of k extremal fundamental loop weight modules $V(Y_{\ell,a_i}Y^{-1}_{0,a_iq^\ell})$ with generic non-zero complex parameters $a_1, a_2, \cdots, a_k \in \mathbb{C}^*$ (Theorem 4.4.3): it is an extremal loop weight module of ℓ-weight

$$Y_{\ell,a_1} \cdots Y_{\ell,a_k} Y^{-1}_{0,a_1q^\ell} \cdots Y^{-1}_{0,a_kq^\ell}.$$

In the third subsection we treat the case of fusion products of extremal fundamental loop weight modules when parameters are chosen non-generic. More precisely we consider the fusion product of vector representations $V(Y_{1,a}Y^{-1}_{0,aq})$ when the set of parameters forms a q-segment $\{a, aq^{-2}, \cdots, aq^{-2(k-1)}\}$ ($k \in \mathbb{N}^*$, $a \in \mathbb{C}^*$). We recover in that way all the extremal fundamental loop weight modules (Theorem 4.4.6). Furthermore we obtain also new extremal loop weight modules (Theorem 4.4.9) of ℓ-weight

$$Y_{\ell,aq^{-\ell+1}} Y_{\ell,aq^{-\ell+3}} \cdots Y_{\ell,aq^{-\ell+1+2k}} Y^{-1}_{0,aq} Y^{-1}_{0,aq^3} \cdots Y^{-1}_{0,aq^{1+2k}}.$$

4.4.1 Existence conditions

Set $\ell \geq 1$. Let us consider the tensor product

$$V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell}) \text{ with } a, b \in \mathbb{C}^*.$$

We determine when the action of $\mathcal{U}_q(\hat{sl}_\infty)$ is well-defined on it.

Proposition 4.4.1. *(i) The action of $\mathcal{U}_q(\hat{sl}_\infty)$ is well-defined on the tensor product $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$ if and only if*

$$\frac{a}{b} \notin \{q^{-2\ell+2}, q^{-2\ell+4}, \cdots, q^{2\ell-2}\}.$$

(ii) Assume that $\frac{a}{b} = q^{-2\ell}$. The module $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$ has a submodule spanned by vectors of the form

$$T_a \otimes T'_b \text{ with } i_1 \leq j_\ell$$

where $T = (i_1 < \cdots < i_\ell)$ and $T' = (j_1 < \cdots < j_\ell)$. The submodule and the quotient module are irreducible and thin.

(iii) Assume that $\frac{a}{b} = q^{2\ell}$. The module $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$ has a submodule spanned by vectors of the form

$$T_a \otimes T'_b \text{ with } i_\ell < j_1$$

where $T = (i_1 < \cdots < i_\ell)$ and $T' = (j_1 < \cdots < j_\ell)$. The submodule and the quotient module are irreducible and thin.

(iv) In all other cases, $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$ *is a thin, irreducible* $\mathcal{U}_q(\hat{sl}_\infty)$*-module.*

Remark 4.4.2. *This result generalizes the one for the (specialized) vector representations of* $\mathcal{U}_q(sl^{tor}_{n+1})$ $(n \geq 2)$ *given in [FJMM13, Man13a].*

Proof. Let $T, T' \in \mathcal{T}_{[1,\ell]}$ and $a, b \in \mathbb{C}^*$. We determine the action of $x_i^-(z)$ on $T_a \otimes T'_b \in V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$: it is equal to

$$\begin{gathered}\sum_{1\leq s\leq \ell} \delta_{\{j_s=i\}}\delta(bq^{i+\ell+1-2s}) \cdot T_a \otimes (\tilde{f}_i \cdot T')_b + \sum_{1\leq r,s\leq \ell} \delta_{\{i_r=i\}}\delta(aq^{i+\ell+1-2r}) \\ \times \Big(1 + \delta_{\{j_s=i+1\}}\delta_{\{j_{s-1}\neq i\}} \left(\psi(\tfrac{b}{a}q^{2(r-s)+2})^{-1} - 1\right) + \delta_{\{j_s=i\}}\delta_{\{j_{s+1}\neq i+1\}} \left(\psi(\tfrac{b}{a}q^{2(r-s)}) - 1\right)\Big) \\ \times (\tilde{f}_i \cdot T)_a \otimes T'_b.\end{gathered}$$

So the action of the operators $x_i^-(z)$ is well-defined on $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$ if and only if

$$\frac{a}{b} \neq q^{2(r-s)} \text{ for all } 1 \leq r, s \leq \ell$$

or in an equivalent way, $\dfrac{a}{b} \notin \{q^{-2\ell+2}, q^{-2\ell+4}, \cdots, q^{2\ell-2}\}$. We proceed in the same way for the $x_i^+(z)$. We obtain the first assertion by Lemma 4.2.22.

Now assume that $\dfrac{a}{b} = q^{-2\ell}$. Let us consider the subvector space V of

$$V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$$

spanned by vectors of the form

$$T_a \otimes T'_b \text{ with } i_1 \leq j_\ell$$

where $T = (i_1 < \cdots < i_\ell)$ and $T' = (j_1 < \cdots < j_\ell)$. Let $i \in \mathbb{Z}$ and $T = (i < i_2 < \cdots < i_\ell)$, $T' = (j_1 < \cdots < j_{\ell-1} < i)$ be semi-standard tableaux. We have

$$\begin{aligned} x_i^-(z) \cdot T_a \otimes T'_b &= \delta(aq^{i+\ell+1}z) T_a \otimes (\tilde{f}_i \cdot T')_b + \delta(aq^{i+\ell-1}z)\psi(q^2)(\tilde{f}_i \cdot T)_a \otimes T'_b \\ &= \delta(aq^{i+\ell+1}) T_a \otimes (\tilde{f}_i \cdot T')_b, \\ x_{i-1}^+(z) \cdot T_a \otimes T'_b &= \delta(aq^{i+\ell-1}z)(\tilde{e}_{i-1} \cdot T)_a \otimes T'_b + \delta(aq^{i+\ell+1}z)\psi(1)^{-1} T_a \otimes (\tilde{e}_{i-1} \cdot T')_{aq^{2\ell}} \\ &= \delta(aq^{i+\ell-1}z)(\tilde{e}_{i-1} \cdot T)_a \otimes T'_b. \end{aligned}$$

By these computations, V is a sub-$\mathcal{U}_q(\hat{sl}_\infty)$-module of $V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}) \otimes V(Y_{\ell,b}Y^{-1}_{0,bq^\ell})$. We proceed in the same way for the third point. The thin property and the irreducibility of the modules in *(ii)*, *(iii)* and *(iv)* follow by straightforward computations. □

4.4.2 The generic case

Let us give a first result about the tensor product of extremal fundamental loop weight modules when the non-zero complex parameters are chosen generics.

Theorem 4.4.3. *Let* $\ell \geq 1$, $k \in \mathbb{N}^*$ *and* $a_1, \cdots, a_k \in \mathbb{C}^*$ *be such that* $\frac{a_i}{a_j} \neq 1$ *and* $\frac{a_i}{a_j} \neq q^{\pm 2}$ *for all* $i < j$*. Then the tensor product of extremal fundamental loop weight modules*

$$V = V(Y_{\ell,a_1}Y^{-1}_{0,a_1q^\ell}) \otimes V(Y_{\ell,a_2}Y^{-1}_{0,a_2q^\ell}) \otimes \cdots \otimes V(Y_{\ell,a_k}Y^{-1}_{0,a_kq^\ell})$$

is an irreducible extremal loop weight module of $\mathcal{U}_q(\hat{sl}_\infty)$ *of* ℓ*-weight*

$$Y_{\ell,a_1} \cdots Y_{\ell,a_k} Y^{-1}_{0,a_1q^\ell} \cdots Y^{-1}_{0,a_kq^\ell}.$$

Proof. As a direct consequence of Proposition 4.4.1, the $\mathcal{U}_q(\hat{sl}_\infty)$-module V is irreducible. Furthermore as $V(Y_{\ell,a_j}Y_{0,a_jq^\ell})$ are extremal loop weight modules for all $1 \leq j \leq k$, there tensor product V satisfies the third condition of Definition 4.2.20. Eventually for $T \in \mathcal{T}_{[1,\ell]}$, set $T' = \tilde{f}_i \cdot T$ and $T'' = \tilde{e}_i \cdot T$. We have for all $i \in \mathbb{Z}$ (see the proof of [Man13a, Theorem 5.3])

$$(x_{i,0}^-)^{(k)} \cdot T_{a_1} \otimes \cdots \otimes T_{a_k} = T'_{a_1} \otimes \cdots \otimes T'_{a_k}$$

and

$$(x_{i,0}^+)^{(k)} \cdot T_{a_1} \otimes \cdots \otimes T_{a_k} = T''_{a_1} \otimes \cdots \otimes T''_{a_k}.$$

The extremality of $(1 < \cdots < \ell)_{a_1} \otimes \cdots \otimes (1 < \cdots < \ell)_{a_k}$ follows. $\square$

4.4.3 The non-generic case

Let us consider the extremal fundamental loop weight module $V(Y_{1,a}Y_{0,aq}^{-1})$ $(a \in \mathbb{C}^*)$ of $\mathcal{U}_q(\hat{sl}_\infty)$. This module has a basis labelled by the tableaux

$$T_a = \boxed{j}_a \text{ with } j \in \mathbb{Z}.$$

Furthermore the action of $\mathcal{U}_q(\hat{sl}_\infty)$ is known, given by

$$\begin{aligned}
x_i^+(z) \cdot \boxed{j}_a &= \delta_{i,j-1}\delta(aq^{j-1}z) \cdot \boxed{j-1}_a, \\
x_i^-(z) \cdot \boxed{j}_a &= \delta_{i,j}\delta(aq^{j}z) \cdot \boxed{j+1}_a, \\
\phi_i^\pm(z) \cdot \boxed{j}_a &= \begin{cases} \psi(aq^j z) \cdot \boxed{j}_a & \text{if } i = j, \\ \psi(aq^{j+1}z)^{-1} \cdot \boxed{j}_a & \text{if } i = j-1, \\ \boxed{j}_a & \text{otherwise.} \end{cases}
\end{aligned}$$

We denote this module $V(\boxed{1}_a)$ in the following. It is defined in [FJMM13, Man12c, Man13a] with different methods for the quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{tor})$ $(n \geq 2)$ and called (specialized) *vector representation*.

For all $k \in \mathbb{N}^*$ and $a \in \mathbb{C}^*$, we consider the tensor product of vector representations

$$V(\boxed{1}_a) \otimes V(\boxed{1}_{aq^{-2}}) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}}).$$

By Proposition 4.4.1, the coproduct Δ_D endows this tensor product with a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module. Let us give a first result about it (the proof is exactly the same as Theorem 3.3.5 and is not recalled).

Theorem 4.4.4. *The $\mathcal{U}_q(\hat{sl}_\infty)$-module $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$ is an extremal loop weight module of ℓ-weight*

$$Y_{1,a}Y_{1,aq^{-2}} \cdots Y_{1,aq^{-2(k-1)}}Y_{0,aq}^{-1}Y_{0,aq^{-1}}^{-1} \cdots Y_{0,aq^{-2(k-1)+1}}^{-1}.$$

Denote by $V\begin{pmatrix} \boxed{1} \\ \vdots \\ \boxed{k}_a \end{pmatrix}$ the subvector space of $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(k-1)}})$ generated by

$$\boxed{i_1}_a \otimes \boxed{i_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{i_k}_{aq^{-2(k-1)}} \text{ with } i_1 < i_2 < \cdots < i_k.$$

Proposition 4.4.5. *The coproduct Δ_D endows the vector space $V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{k}_a\end{matrix}\right)$ with a structure of thin $\mathcal{U}_q(\hat{sl}_\infty)$-module.*

Proof. The action of the $x_i^{\pm}(z)$ is well-defined on it by Proposition 4.4.1. It remains to show that for a vector

$$v = \boxed{i_1}_a \otimes \boxed{i_2}_{aq^{-2}} \otimes \cdots \otimes \boxed{i_k}_{aq^{-2(k-1)}} \text{ with } i_1 < i_2 < \cdots < i_k,$$

$x_i^{\pm}(z) \cdot v \in V$ for all $i \in \mathbb{Z}$. This is a consequence of the equalities

$$\begin{aligned} x_i^+(z) \cdot \boxed{i}_a \otimes \boxed{i{+}1}_{aq^{-2}} &= \delta(aq^{i-2}z)\psi(q^2) \cdot \boxed{i}_a \otimes \boxed{i}_{aq^{-2}} = 0, \\ x_i^-(z) \cdot \boxed{i}_a \otimes \boxed{i{+}1}_{aq^{-2}} &= \delta(aq^{i}z)\psi(1)^{-1} \cdot \boxed{i{+}1}_a \otimes \boxed{i{+}1}_{aq^{-2}} = 0. \end{aligned} \tag{4.13}$$

Eventually these modules are thin, the weight of the vector $\boxed{i_1}_a \otimes \cdots \otimes \boxed{i_k}_{aq^{-2(k-1)}}$ being completely determined by the sequence $i_1 < i_2 < \cdots < i_k$. □

Theorem 4.4.6. *The $\mathcal{U}_q(\hat{sl}_\infty)$-module $V = V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{\ell}\end{matrix}\right)_a$ ($\ell \geq 1, a \in \mathbb{C}^*$) is isomorphic to the extremal fundamental loop weight module $V(Y_{\ell,aq^{-\ell+1}}Y_{0,aq}^{-1})$.*

Proof. Let us define the morphism of vector spaces $f : V(Y_{\ell,aq^{-\ell+1}}Y_{0,aq}) \to V$ by setting

$$f(T) = \boxed{i_1}_a \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_\ell}_{aq^{-2(\ell-1)}}$$

for all semi-standard tableaux $T = (i_1 < i_2 < \cdots < i_\ell)$ in $\mathcal{T}_{[1,\ell]}$. Hence defined f is an isomorphism of $\mathcal{U}_q(\hat{sl}_\infty)$-modules: this follows by straightforward computations. □

Proposition 4.4.7. *Fix $a \in \mathbb{C}^*, k \in \mathbb{N}^*$ and $\ell_1, \ell_2, \cdots, \ell_k \geq 1$. Then Δ_D endows the tensor product*

$$V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{\ell_1}\end{matrix}\right)_a \otimes V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{\ell_2}\end{matrix}\right)_{aq^{-2\ell_1}} \otimes \cdots \otimes V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{\ell_k}\end{matrix}\right)_{aq^{-2(\ell_1+\cdots+\ell_{k-1})}}$$

with a structure of $\mathcal{U}_q(\hat{sl}_\infty)$-module. Furthermore, it has a submodule isomorphic to

$$V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{\ell}\end{matrix}\right)_a \text{ with } \ell = \ell_1 + \ell_2 + \cdots + \ell_k.$$

Proof. By Proposition 4.4.1, the action on the tensor product is well-defined. Furthermore let us consider the subvector space generated by vectors of the form

$$\boxed{i_1^{(1)}} \otimes \cdots \otimes \boxed{i_{\ell_1}^{(1)}} \otimes \boxed{i_1^{(2)}} \otimes \cdots \otimes \boxed{i_{\ell_2}^{(2)}} \otimes \cdots \otimes \boxed{i_1^{(k)}} \otimes \cdots \otimes \boxed{i_{\ell_k}^{(k)}}$$

with $i_1^{(1)} < \cdots < i_{\ell_1}^{(1)} < i_1^{(2)} < \cdots < i_{\ell_2}^{(2)} < \cdots < i_1^{(k)} < \cdots < i_{\ell_k}^{(k)}$. It is by definition the submodule $V\left(\begin{matrix}\boxed{1}\\ \vdots \\ \boxed{\ell}\end{matrix}\right)_a$ of $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(\ell-1)}})$ with $\ell = \ell_1 + \cdots + \ell_k$. □

Set $\ell, k \geq 1$ and $a \in \mathbb{C}^*$. Let us consider the tensor product V of vector representations

$$V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(\ell-1)}}) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(1-k)}}) \otimes \cdots \otimes V(\boxed{1}_{aq^{-2(\ell-k)}}).$$

Let $\mathcal{T}_{[1,\ell]\times[1,k]}$ be the set of semi-standard tableaux $T = (T_{i,j})_{(i,j)\in[1,\ell]\times[1,k]}$. Then $(\mathcal{T}_{[1,\ell]\times[1,k]}, \mathrm{wt}, \tilde{e}_i, \tilde{f}_i)$ is a $\mathcal{U}_q(sl_\infty)$-crystal [Kas02a]. To a tableaux $T = (T_{i,j}) \in \mathcal{T}_{[1,\ell]\times[1,k]}$, it corresponds a vector T_a in V, defined by (we read the tableaux from top to bottom and from left to right)

$$T_a = \boxed{T_{1,1}}_a \otimes \cdots \otimes \boxed{T_{\ell,1}}_{aq^{-2(\ell-1)}} \otimes \cdots \otimes \boxed{T_{1,k}}_{aq^{-2(1-k)}} \otimes \cdots \otimes \boxed{T_{\ell,k}}_{aq^{-2(\ell-k)}}.$$

Denote by $T^e = (T_{i,j})$ the semi-standard tableaux such that $T_{i,j} = i$ for all $(i,j) \in [1,\ell] \times [1,k]$:

$$T^e = \begin{array}{|c|c|c|}\hline 1 & \dots & 1 \\ \hline \vdots & & \vdots \\ \hline \ell & \dots & \ell \\ \hline \end{array}.$$

Let $V(T^e_a)$ be the subvector space of V generated by vectors T_a with $T \in \mathcal{T}_{[1,\ell]\times[1,k]}$.

Proposition 4.4.8. *Set $\ell, k \geq 1$ and $a \in \mathbb{C}^*$. The coproduct Δ_D endows $V(T^e_a)$ with a structure of thin $\mathcal{U}_q(\hat{sl}_\infty)$-module.*

Proof. Let $T = (T_{i,j}) \in \mathcal{T}_{[1,\ell]\times[1,k]}$ be a semi-standard tableaux. We have to show that for all $s \in \mathbb{Z}$, $x_s^-(z) \cdot T_a$ is also in $V(T^e_a)$. Actually it suffices to consider the following cases:

- if T is such that there exist $1 \leq i \leq \ell - 1$ and $1 \leq j \leq k$ such that

$$T_{i,j} = s \text{ and } T_{i,j+1} = s + 1.$$

  Then by (4.13), $x_s^-(z) \cdot T_a$ is in $V(T^e_a)$.
- if T is such that there exist $1 \leq i \leq \ell$ and $p \geq 1$, $1 \leq j \leq k - p$ such that

$$T_{i,j} = T_{i,j+1} = \cdots = T_{i,j+p} = s \text{ and } T_{i,j+p+1} \geq s + 1.$$

  Then the fact that $x_s^-(z) \cdot T_a$ belongs to $V(T^e_a)$ is a consequence of the following equality in $V(\boxed{1}_a) \otimes \cdots \otimes V(\boxed{1}_{aq^{2p}})$

$$x_s^-(z) \cdot \boxed{s}_a \otimes \cdots \otimes \boxed{s}_{aq^{2p}} = \delta(aq^{s+2p}z) \boxed{s}_a \otimes \cdots \otimes \boxed{s}_{aq^{2(p-1)}} \otimes \boxed{s{+}1}_{aq^{2p}}.$$

We proceed in the same way for the operators $x_s^+(z)$. $\square$

Theorem 4.4.9. *Set $\ell, k \geq 1$ and $a \in \mathbb{C}^*$. Then $V(T^e_a)$ is an irreducible extremal loop weight $\mathcal{U}_q(\hat{sl}_\infty)$-module of ℓ-weight*

$$m_{T^e} = Y_{\ell,aq^{-\ell+1}} Y_{\ell,aq^{-\ell+3}} \cdots Y_{\ell,aq^{-\ell+1+2(k-1)}} Y^{-1}_{0,aq} Y^{-1}_{0,aq^3} \cdots Y^{-1}_{0,aq^{1+2(k-1)}}.$$

Proof. Let us recall that the semi-standard tableaux

$$T^e = \begin{array}{|c|c|c|}\hline 1 & \dots & 1 \\ \hline \vdots & & \vdots \\ \hline \ell & \dots & \ell \\ \hline \end{array}$$

is an extremal element of the $\mathcal{U}_q(sl_\infty)$-crystal $\mathcal{T}_{[1,\ell]\times[1,k]}$ of weight $k\Lambda_\ell - k\Lambda_0$. Furthermore we have

$$W \cdot T^e = \{T = (T_{i,j})_{(i,j)\in[1,\ell]\times[1,k]} | T_{i,1} = T_{i,2} = \cdots = T_{i,k} \text{ for all } 1 \leq i \leq \ell\}.$$

Using the equalities ($s \in \mathbb{Z}$)

$$(x^-_{s,0})^{(k)} \cdot \boxed{s}_a \otimes \cdots \otimes \boxed{s}_{aq^{2(k-1)}} = \boxed{s+1}_a \otimes \cdots \otimes \boxed{s+1}_{aq^{2(k-1)}},$$
$$(x^+_{s-1,0})^{(k)} \cdot \boxed{s}_a \otimes \cdots \otimes \boxed{s}_{aq^{2(k-1)}} = \boxed{s-1}_a \otimes \cdots \otimes \boxed{s-1}_{aq^{2(k-1)}},$$

we get

$$x_i^\pm \cdot T_a = 0 \text{ and } (x_i^\mp)^{(\pm \mathrm{wt}(T)(h_i))} \cdot T_a = S_i(T)_a \text{ if } \pm \mathrm{wt}(T)(h_i) \geq 0$$

for all $T \in W \cdot T^e$. Then the result is a consequence of the following Lemma. □

Lemma 4.4.10. *Let V be a $\mathcal{U}_q(sl_\infty)$-module with basis $(v_b)_{b\in\mathcal{B}}$ indexed by a $\mathcal{U}_q(sl_\infty)$-crystal $\mathcal{B}$. Assume that $b^e \in \mathcal{B}$ is extremal of weight $\lambda \in P$ and for all $i \in \mathbb{Z}$ and $b \in W \cdot b^e$,*

$$\mathrm{wt}(v_b) = \mathrm{wt}(b),\ x_i^\pm \cdot v_b = 0 \textit{ and } (x_i^\mp)^{(\pm\mathrm{wt}(b)(h_i))} \cdot v_b = v_{S_i(b)} \textit{ if } \pm \mathrm{wt}(b)(h_i) \geq 0.$$

Then v_{b^e} is an extremal vector of weight λ.

Proof. The proof is analogue to the one of Lemma 4.3.13. □

4.5 Application to quantum toroidal algebras

The quantum toroidal algebras $\mathcal{U}_q(sl^{tor}_{n+1})$ ($n \geq 2$) were introduced by Ginzburg-Kapranov-Vasserot in type A [GKV95]. They are quantum groups analogs of elliptic Cherednik algebras [Che95] to whom they are related via Schur-Weyl duality [VV96]. Extremal loop weight modules have been defined in the context of $\mathcal{U}_q(sl^{tor}_{n+1})$ in Chapter 2 and Chapter 3. The main motivation is the construction of finite-dimensional representations of the quantum toroidal algebras at roots of unity.

A combinatorial link between the representation theory of $\mathcal{U}_q(\hat{sl}_\infty)$ and of quantum toroidal algebras is conjectured in [Her11]. This conjecture is proved for the class of Kirillov-Reshetikhin modules. The main motivation in [Her11] is to predict q–character formulae for representations of $\mathcal{U}_q(sl^{tor}_{n+1})$.

In this section, we prove [Her11, Conjecture 5.3] for the particular family of extremal fundamental loop weight modules of $\mathcal{U}_q(\hat{sl}_\infty)$: by the combinatorial link highlighted in [Her11], they are related to $\mathcal{U}_q(sl^{tor}_{n+1})$-modules defined in Chapter 3. The aim is to construct extremal loop weight modules of $\mathcal{U}_q(sl^{tor}_{n+1})$. We show that we recover the extremal fundamental loop weight modules constructed in Chapter 2 and Chapter 3.

In the first subsection, we recall some definitions about the quantum toroidal algebras $\mathcal{U}_q(sl^{tor}_{n+1})$ ($n \geq 2$) and its combinatorial link with $\mathcal{U}_q(\hat{sl}_\infty)$ conjectured in [Her11].

In the second subsection, we prove this conjecture for the class of extremal fundamental loop weight modules of $\mathcal{U}_q(\hat{sl}_\infty)$ (Theorem 4.5.3). We recover in that way the extremal fundamental loop weight modules of $\mathcal{U}_q(sl^{tor}_{n+1})$ constructed in Chapter 3 (Theorem 4.5.4).

4.5.1 Reminder about quantum toroidal algebras

Let us recall the definition of the quantum toroidal algebra $\mathcal{U}_q(sl^{tor}_{n+1})$ ($n \geq 2$). Set $I_n = \mathbb{Z}/(n+1)\mathbb{Z}$. Let $C = (C_{\bar{i},\bar{j}})_{\bar{i},\bar{j}\in I_n}$ be the Cartan matrix of type $A_n^{(1)}$,

$$C_{\bar{i},\bar{i}} = 2\ ,\ C_{\bar{i},\overline{i+1}} = C_{\overline{i+1},\bar{i}} = -1 \text{ and } C_{\bar{i},\bar{j}} = 0 \text{ if } \bar{j} \neq \bar{i}, \overline{i \pm 1}.$$

The algebra $\mathcal{U}_q(sl_{n+1}^{tor})$ is defined by the same generators and relations as in Definition 4.2.10 with $\bar{i}, \bar{j} \in I_n$.

The representation theory of $\mathcal{U}_q(sl_{n+1}^{tor})$ is similar to the one of $\mathcal{U}_q(\hat{sl}_\infty)$: the simple integrable ℓ-highest weight modules are parametrized by Drinfeld polynomials. In particular, the Kirillov-Reshetikhin modules can be defined in an analogue way. One defines q–characters $\chi_{q,n}(V)$ as above for integrable representations V with finite-dimensional ℓ-weight spaces: it is a sum of elements in A_n, where A_n is the group of monomials of the form

$$m = \prod_{\bar{i} \in I_n, a \in \mathbb{C}^*} Y_{\bar{i},a}^{u_{\bar{i},a}(m)}, \ u_{\bar{i},a}(m) \in \mathbb{Z}.$$

Consider the ring morphism

$$\phi_n : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in \mathbb{Z}, a \in \mathbb{C}^*} \to \mathbb{Z}[Y_{\bar{i},a}^{\pm 1}]_{\bar{i} \in I_n, a \in \mathbb{C}^*}$$

defined, for $i \in \mathbb{Z}$ and $a \in \mathbb{C}^*$, by

$$\phi_n(Y_{i,a}^{\pm 1}) = Y_{\bar{i},a}^{\pm 1}.$$

Let $\mathrm{Im}(\chi_q)$ (resp. $\mathrm{Im}(\chi_{q,n})$) be the image of the Grothendieck group corresponding to the category of integrable $\mathcal{U}_q(\hat{sl}_\infty)$-modules (resp. integrable $\mathcal{U}_q(sl_{n+1}^{tor})$-modules) belonging to the category $\mathcal{O}$. The morphism ϕ_n gives rise naturally to a group morphism

$$\phi_n : \mathrm{Im}(\chi_q) \to \mathbb{Z}[[Y_{\bar{i},a}^{\pm 1}]]_{\bar{i} \in I_n, a \in \mathbb{C}^*}.$$

More precisely, one can show that $\phi_n\left(\mathrm{Im}(\chi_q)\right) \subset \mathrm{Im}(\chi_{q,n})$ by using the characterization stated in the proof of [Her11, Theorem 4.2]. Hence for V a $\mathcal{U}_q(\hat{sl}_\infty)$-module in the category $\mathcal{O}_{\mathrm{int}}$, $\phi_n\left(\chi_q(V)\right)$ is the q–character of a representation of $\mathcal{U}_q(sl_{n+1}^{tor})$, which is a priori virtual. It is conjectured in [Her11] that

Conjecture 4.5.1. *Let V be a simple representation in $\mathcal{O}_{\mathrm{int}}$ for $\mathcal{U}_q(\hat{sl}_\infty)$. Then $\phi_n(\chi_q(V))$ is the q–character of an actual representation of $\mathcal{U}_q(sl_{n+1}^{tor})$.*

This conjectural link between the representation theory of $\mathcal{U}_q(\hat{sl}_\infty)$ and the one of quantum toroidal algebras is proved in [Her11] for Kirillov-Reshetikhin modules. In the following section, we give one of the main results of this chapter: we prove Conjecture 4.5.1 in the context of extremal fundamental loop weight modules of $\mathcal{U}_q(\hat{sl}_\infty)$.

4.5.2 Extremal loop weight modules for quantum toroidal algebras

Let us consider the extremal fundamental loop weight module $V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})$ with $\ell \geq 1$ and $a \in \mathbb{C}^*$. Recall that we have determined its q–character: it is given by

$$\chi_q(V(Y_{\ell,a}Y_{0,aq^\ell}^{-1})) = \sum_{T \in \mathcal{T}_{[1,\ell]}} m_T$$

where $\mathcal{T}_{[1,\ell]}$ is the set of semi-standard tableaux of shape (ℓ) and

$$m_T = \prod_{1 \leq j \leq \ell} \boxed{i_j}_{\,aq^{\ell+1-2j}}.$$

Furthermore, we have shown that

$$\{m_T | T \in \mathcal{T}_{[1,\ell]}\} = \mathcal{M}(Y_{\ell,a}Y_{0,aq^\ell}^{-1}).$$

Let us consider the ring morphism $\phi_n : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in\mathbb{Z},a\in\mathbb{C}^*} \to \mathbb{Z}[Y_{\bar{i},a}^{\pm 1}]_{\bar{i}\in I_n,a\in\mathbb{C}^*}$. It cannot be extend to an application $\mathbb{Z}^A \to \mathbb{Z}^{A_n}$. However, ϕ_n gives rise to a well-defined application

$$\phi_n : \mathbb{Z}^{\mathcal{M}(Y_{\ell,a}Y^{-1}_{0,aq^\ell})} \longrightarrow \mathbb{Z}^{A_n}.$$

Actually by definition of the monomial crystal $\mathcal{M}$ (see [Kas03, Man12c, Nak03]), ϕ_n can be defined in all its connected components.

Proposition 4.5.2. *$\phi_n\left(\chi_q(V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}))\right)$ is the q–character of a virtual representation of $\mathcal{U}_q(sl^{tor}_{n+1})$.*

Proof. We have to check that $\phi_n\left(\chi_q(V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}))\right)$ respects the following characterization (see [Her11, Theorem 4.2])

it is an infinite sum of elements in $\mathbb{Z}[Y_{\bar{r},a}(1+A^{-1}_{\bar{r},aq}), Y^{\pm 1}_{\bar{r'},a}]_{a\in\mathbb{C}^*,\bar{r}\neq\bar{r'}}$ for each $\bar{r}\in I_n$.

Actually, it suffices to check the analogue property at the level of the $\mathcal{U}_q(sl_\infty)$-crystal $\mathcal{M}(Y_{\ell,a}Y^{-1}_{0,aq^\ell})$: this holds by the description of it given above (see Remark 4.3.7). ☐

Let us give one of the main results of this chapter.

Theorem 4.5.3. *Assume that $n \geq 2$ is such that*

$$(n \text{ is even and } \ell \leq n+1) \text{ or } \left(n \text{ is odd and } \ell \leq \frac{n+1}{2}\right). \tag{4.14}$$

Then $\phi_n\left(\chi_q(V(Y_{\ell,a}Y^{-1}_{0,aq^\ell}))\right)$ is the q–character of an actual representation of $\mathcal{U}_q(sl^{tor}_{n+1})$. We denote it $V_n(Y_{\ell,a}Y^{-1}_{0,aq^\ell})$.

This result confirms Conjecture 4.5.1 in the context of extremal fundamental loop weight modules.

Proof. This is a consequence of Proposition 3.3.4. In fact if (4.14) holds, we have shown that there exists a sub-$\mathcal{U}_q(sl^{tor}_{n+1})$-module [2] $V_n(Y_{\ell,a}Y^{-1}_{0,aq^\ell})$ of the fusion product of specialized vector representations with basis labelled by the set $\mathcal{T}_{[1,\ell]}$ of semi-standard tableaux of shape (ℓ). Furthermore for all $T\in\mathcal{T}_{[1,\ell]}$, T_a is an ℓ-weight vector of ℓ-weight $\phi_n(m_T)$. The result follows directly. ☐

The main motivation of this study is the construction of extremal loop weight modules for $\mathcal{U}_q(sl^{tor}_{n+1})$. Actually we have

Theorem 4.5.4. *[Man13a]*

- *(i) Assume that $n \geq 2$. Then $V_n(Y_{1,a}Y^{-1}_{0,aq})$ is an extremal loop weight module of ℓ-weight $Y_{1,a}Y^{-1}_{0,aq}$.*
- *(ii) Assume that $n = 2r+1$ is odd ($r \geq 1$). Then $V_n(Y_{r+1,a}Y^{-1}_{0,aq^{r+1}})$ admits an irreducible quotient which is an extremal loop weight module of ℓ-weight $Y_{r+1,a}Y^{-1}_{0,aq^{r+1}}$.*

Remark 4.5.5. *In the other cases, the representations $V_n(Y_{\ell,a}Y^{-1}_{0,aq^\ell})$ are not ℓ-extremal (see Chapter 3).*

2. It is denoted $\tilde{V}\left(\begin{array}{c}\boxed{1}\\ \vdots \\ \boxed{k}\end{array}\right)_{aq^{\ell-1}}$ in Chapter 3.

Chapter 5

Further possible developments

In this chapter we construct an extremal loop weight module for the quantum toroidal algebra of type D_4 with the features we developed in Chapter 2 and Chapter 3. This is the first example of such module in type different to A. And one can sensibly attend to get analogue results for general quantum affinizations in further developments.

In the first section, we construct the fusion product

$$V(e^{\Lambda_1}Y_{1,a}) \otimes V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$$

of the simple ℓ-highest weight module $V(e^{\Lambda_1}Y_{1,a})$ and the simple ℓ-lowest weight module $V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$ ($a \in \mathbb{C}^*$) (Proposition 5.1.1).

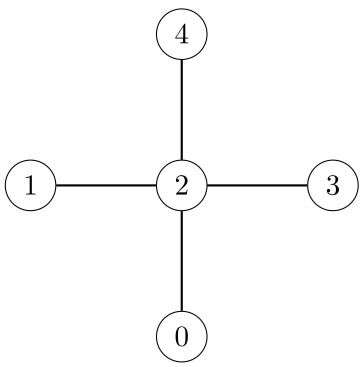
We study this fusion product in the second section. In particular we show that we obtain in that way an extremal loop weight $\mathcal{U}_q(\mathfrak{g}^{tor})$-module (Theorem 5.1.2). By specializing the quantum parameter, we get finite-dimensional representations of the quantum toroidal algebra of type D_4 at roots of unity (Proposition 5.1.3).

In the third section, we discuss further possible developments of this thesis.

5.1 Extremal loop weight modules in type D_4

5.1.1 Fusion product construction

We use the fusion product process developed in Section 3.2 to construct extremal loop weight modules by tensor product for $\mathcal{U}_q(\mathfrak{g}^{tor})$ with $\mathfrak{g}$ the simple Lie algebra of type D_4 associated to the Dynkin diagram D.



We consider the tensor product

$$V(e^{\Lambda_1}Y_{1,a}) \otimes V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$$

of the simple ℓ-highest weight module $V(e^{\Lambda_1}Y_{1,a})$ and the simple ℓ-lowest weight module $V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$ ($a \in \mathbb{C}^*$).

Proposition 5.1.1. *The coproduct Δ_D endows $V(e^{\Lambda_1}Y_{1,a}) \otimes V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$ with a structure of $\mathcal{U}_q(\mathfrak{g}^{tor})$-module.*

Proof. Fix $i \in I$ and let us show that the action of the operator $x_i^+(z)$ is well-defined on this tensor product (we proceed in the same way for $x_i^-(z)$). The proof is similar to the one of Theorem 3.2.1. Actually we have to show that the infinite sum (3.5) is well-defined in our situation. For that we study the q–characters of $V(e^{\Lambda_1}Y_{1,a})$ and $V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$. They can be computed and we check that in the sum (3.5),

$$\begin{aligned}
&\lambda_- \in \frac{1}{a}(\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) \text{ and } \lambda_+ \in a(-\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) && \text{if } i = 1,\\
&\lambda_- \in \frac{1}{aq}(\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) \text{ and } \lambda_+ \in aq(-\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) && \text{if } i = 2,\\
&\lambda_- \in \frac{1}{aq^2}(\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) \text{ and } \lambda_+ \in a(-\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) && \text{if } i = 3, 4,\\
&\lambda_- \in \frac{1}{aq^2}(\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) \text{ and } \lambda_+ \in aq^2(-\mathbb{N} + q^{-1}\mathbb{Z}[q^{-1}]) && \text{if } i = 0,
\end{aligned}$$

and $\lambda_- \cdot \lambda_+ \neq 1$ in all the situations. Hence the action of $x_i^{\pm}(z)$ is well-defined on $V(e^{\Lambda_1}Y_{1,a}) \otimes V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$. Then $V(e^{\Lambda_1}Y_{1,a}) \otimes V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$ can be endowed with a structure of $\mathcal{U}_q(\mathfrak{g}^{tor})$-module by Lemma 1.8.1. □

5.1.2 Study of $T(e^{\varpi_1}Y_{1,1}Y_{0,q^2}^{-1})$

Let $T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})$ be the sub-$\mathcal{U}_q(\mathfrak{g}^{tor})$-module of $V(e^{\Lambda_1}Y_{1,a}) \otimes V(e^{-\Lambda_0}Y_{0,aq^2}^{-1})$ generated by $v = v^+ \otimes v^-$ where v^+ (resp. v^-) is the ℓ-highest weight vector of $V(e^{\Lambda_1}Y_{1,a})$ (resp. the ℓ-lowest weight vector of $V(e^{-\Lambda_0}Y_{0,aq^2})$).

Theorem 5.1.2. *The $\mathcal{U}_q(\mathfrak{g}^{tor})$-module $T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})$ is an extremal loop weight module generated by the vector v of ℓ-weight $e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1}$.*

Proof. The $\mathcal{U}_q(\mathfrak{g}^{tor})$-module $T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})$ is cyclic and integrable (Lemma 1.8.3). Let us show the third point of Definition 1.7.1. For that, denote by θ the automorphism of the Dynkin diagram D corresponding to a rotation, such that

$$\theta(2) = 2,\ \theta(1) = 0,\ \theta(0) = 3,\ \theta(3) = 4.$$

It defines naturally an automorphism (also denoted by θ) of the quantum toroidal algebra $\mathcal{U}_q(\mathfrak{g}^{tor})$. By restriction, it gives an isomorphism of algebras between all the vertical quantum affine subalgebras $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ for $j = 0, 1, 3, 4$. So to prove the third point of Definition 1.7.1, it reduces to show that $\mathcal{U}_q^{v,j}(\mathfrak{g}^{tor}) \cdot w$ is finite-dimensional for all $w \in T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})$ and $j = 0, 2$ (the other cases $j = 1, 3, 4$ can be deduced from the case $j = 0$ by applying θ). We proceed as in the proof of Proposition 1.7.2. The main point is to show that the set $\{\mu \in \mathrm{wt}(V(e^{\Lambda_1}Y_{1,a}))|\mu(d) = k\}$ is finite for all $k \in \mathbb{Z}$. It can be seen by straightforward computations from the q–character of $V(e^{\Lambda_1}Y_{1,a})$. The last case we have to consider is for $j = 2$. We have

$$\mathcal{U}_q^{v,2}(\mathfrak{g}^{tor}) \simeq \mathcal{U}_q(\hat{sl}_2)' \times \mathcal{U}_q(\hat{sl}_2)' \times \mathcal{U}_q(\hat{sl}_2)' \times \mathcal{U}_q(\hat{sl}_2)'.$$

Then in that case it follows directly from the integrability of $T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})$.

It remains to show the extremality of $v = v^+ \otimes v^-$ for the horizontal quantum affine subalgebra $\mathcal{U}_q^h(\mathfrak{g}^{tor})$. One can process as in the Appendix for the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$, by straightforward computations. We do not detail the calculations. □

The $\mathcal{U}_q(\mathfrak{g}^{tor})$-module $T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})$ is the first example of extremal loop weight module in type different to A.

The $\mathcal{U}_q(\mathfrak{g}^{tor})$-module $T(e^{\varpi_1}Y_{1,1}Y_{0,q^2}^{-1})$ can be defined from a monomial realization of $\mathcal{B}(\varpi_1)$ as in Chapter 2. More precisely, consider the monomial $\mathcal{U}_q(\hat{\mathfrak{g}})$-crystal $\mathcal{M}(e^{\varpi_1}Y_{1,1}Y_{0,q^2}^{-1})$. By [HN06], it is isomorphic to $\mathcal{B}(\varpi_1)$. And one can show that the $\mathcal{U}_q(\mathfrak{g}^{tor})$-module defined from this crystal and the formulas (2.8) is isomorphic to $T(e^{\varpi_1}Y_{1,1}Y_{0,q^2}^{-1})$.

As for the extremal fundamental loop weight $\mathcal{U}_q(sl_{n+1}^{tor})$-modules, $T(e^{\varpi_1}Y_{1,1}Y_{0,q^2}^{-1})$ is an irreducible $\mathcal{U}_q(\mathfrak{g}^{tor})$-module. Furthermore, $\mathrm{Res}(T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1}))$ is isomorphic to the extremal fundamental weight module $V(\varpi_1)$ over $\mathcal{U}_q(\hat{\mathfrak{g}})$.

We show that the map $\tau_{6,-\delta} : \mathcal{M}(e^{\varpi_1}Y_{1,1}Y_{0,q^2}^{-1}) \to \mathcal{M}(e^{\varpi_1}Y_{1,1}Y_{0,q^2}^{-1})$ defined by

$$\tau_{6,-\delta}(e^{\lambda}\prod Y_{i,q^n}^{u_{i,n}}) = e^{\lambda-\delta}\prod Y_{i,q^{n+6}}^{u_{i,n}}$$

is a P_{cl}-crystal automorphism. This shift automorphism is related to the existence of finite-dimensional representations of the quantum toroidal algebra $\mathcal{U}_\epsilon(\mathfrak{g}^{tor})'$ at a root of unity ϵ. In fact by specializing q at a root of unity in the representation $T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})$, we show that

Proposition 5.1.3. *Set $L \geq 1$ and assume that ϵ is a primitive $[6L]$-root of unity. There is an irreducible $\mathcal{U}_\epsilon(\mathfrak{g}^{tor})'$-module $T(e^{\varpi_1}Y_{1,a}Y_{0,aq^2}^{-1})_\epsilon$ of dimension $8L$.*

5.2 Further possible developments

In this section, we give some promising directions for further developments of this thesis.

One of possible perspectives is the construction of extremal loop weight modules for general quantum affinizations. Let us give some results which could help us to do it. In the construction done in Chapter 2, monomial realizations of crystals and promotion operators have a crucial role. Furthermore the same ingredients appear in the example above in type D_4. Actually, some related results are also known in other types.

– The monomial realizations of extremal fundamental weight crystals are explicitly described in [HN06] and the automorphisms z_ℓ are determined for all the quantum affine algebras of non exceptional types.
– there exists symmetry property for crystals arising from automorphisms of the associated Dynkin diagram (analogue of promotion operators in type A). Using that, a combinatorial process allows to obtain Kirillov-Reshetikhin crystals from crystals of finite type (see [FOS09, KKM+92, OS08]). This symmetry property will be useful for a similar construction of extremal loop weight modules in other types.

It could also be interesting to discuss the existence and uniqueness of promotion operators for all the extremal weight crystals, in the spirit of [Shi02].

Let us recall that the quantum toroidal algebras $\mathcal{U}_q(sl_{n+1}^{tor})$ are in Schur-Weyl duality with the elliptic Cherednik algebras [VV96]. So another possible direction could be to study the finite-dimensional representations of Cherednick algebras at roots of unity obtained from the finite-dimensional representations of $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$ defined in Chapter 2 and 3.

In a more long term, one of the main perspectives of this thesis is the understanding of the category of finite-dimensional representations of the quantum toroidal algebras at roots of unity. We have obtained here some irreducible finite-dimensional representations of the quantum toroidal algebras. The first and difficult question is to know if there are other ones, or if we get here all the simple objects of this category. As it is already the case for the quantum groups and the quantum affine algebras, the consequences of such results could be numerous with fruitful applications.

Appendix A

Monomial crystals

In this Appendix, we describe the monomial crystal

$$\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{s\in\mathbb{N}} \mathcal{M}(e^{2\varpi_1+s\delta}Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}).$$

More precisely, we represent the two connected components $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and $\mathcal{M}(e^{2\varpi_1+\delta}Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1})$ of $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Recall that all its connected components are isomorphic to each other modulo shift of weight by δ. Furthermore the map $\tau_{4,-2\delta}$ is an automorphism of these crystals and we only give a part of them. The full crystals are obtained by applying the automorphism $\tau_{4,-2\delta}$. The sub-I_0-crystals

$$\mathcal{M}^1_{0,0,0} = \mathcal{M}_{I_0}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$$

and

$$\mathcal{M}_{0,0,1} = \mathcal{M}_{I_0}(Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1}) \oplus \mathcal{M}_{I_0}(Y_{1,1}^{-1}Y_{1,5}Y_{2,0}Y_{0,6}^{-1})$$

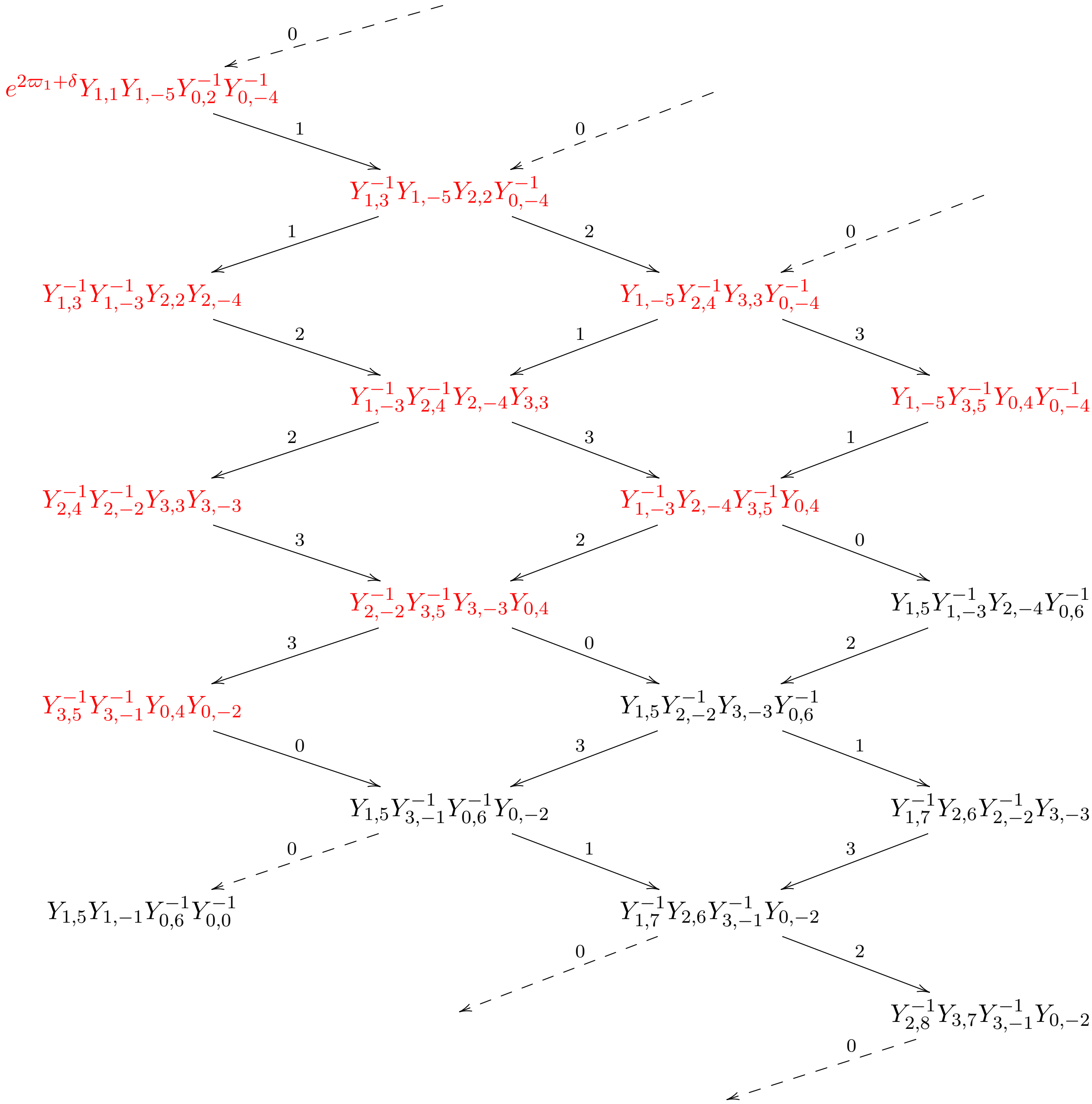
are explicitly given.

Note that the θ-twisted automorphism ϕ of $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ can be viewed as a descent of one diagonal in these crystals.

We refer to [Man12b] for another description of the connected component $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ of $\overline{\mathcal{M}}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$: this monomial crystal is rewritten in another way, and the θ-twisted automorphism ϕ of $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ appears as a rotation. We present also in [Man12b] the link between $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and the $\mathcal{U}_q(\hat{sl}_4)'$-crystal $\mathcal{B}_0(2\Lambda_1)_{\mathrm{aff}}$ (associated affine P_{cl}-crystal obtained from the finite $\mathcal{U}_q(sl_4)$-crystal $\mathcal{B}_0(2\Lambda_1)$ and promotion operator).

$$\begin{tikzcd}
 & {} \arrow[dl, dashed, "0"'] & & \\
e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1} \arrow[dr, "1"] & & {} \arrow[dl, dashed, "0"'] & \\
 & Y_{1,3}^{-1}Y_{1,-1}Y_{2,2}Y_{0,0}^{-1} \arrow[dl, "1"'] \arrow[dr, "2"] & & {} \arrow[dl, dashed, "0"'] \\
Y_{1,3}^{-1}Y_{1,1}^{-1}Y_{2,2}Y_{2,0} \arrow[dr, "2"] & & Y_{1,-1}Y_{2,4}^{-1}Y_{3,3}Y_{0,0}^{-1} \arrow[dl, "1"'] \arrow[dr, "3"] & \\
 & Y_{1,1}^{-1}Y_{2,4}^{-1}Y_{2,0}Y_{3,3} \arrow[dl, "2"'] \arrow[dr, "3"] & & Y_{1,-1}Y_{3,5}^{-1}Y_{0,4}Y_{0,0}^{-1} \arrow[dl, "1"'] \\
Y_{2,4}^{-1}Y_{2,2}^{-1}Y_{3,3}Y_{3,1} \arrow[dr, "3"] & & Y_{1,1}^{-1}Y_{2,0}Y_{3,5}^{-1}Y_{0,4} \arrow[dl, "2"'] \arrow[dr, "0"] & \\
 & Y_{2,2}^{-1}Y_{3,5}^{-1}Y_{3,1}Y_{0,4} \arrow[dl, "3"'] \arrow[dr, "0"] & & Y_{1,5}Y_{1,1}^{-1}Y_{2,0}Y_{0,6}^{-1} \arrow[dl, "2"'] \\
Y_{3,5}^{-1}Y_{3,3}^{-1}Y_{0,4}Y_{0,2} \arrow[dr, "0"] & & Y_{1,5}Y_{2,2}^{-1}Y_{3,1}Y_{0,6}^{-1} \arrow[dl, "3"'] \arrow[dr, "1"] & \\
 & Y_{1,5}Y_{3,3}^{-1}Y_{0,6}^{-1}Y_{0,2} \arrow[dl, dashed, "0"'] \arrow[dr, "1"] & & Y_{1,7}^{-1}Y_{2,6}Y_{2,2}^{-1}Y_{3,1} \arrow[dl, "3"'] \\
Y_{1,5}Y_{1,3}Y_{0,6}^{-1}Y_{0,4}^{-1} & & Y_{1,7}^{-1}Y_{2,6}Y_{3,3}^{-1}Y_{0,2} \arrow[dl, dashed, "0"'] \arrow[dr, "2"] & \\
 & {} & & Y_{2,8}^{-1}Y_{3,7}Y_{3,3}^{-1}Y_{0,2} \arrow[dl, dashed, "0"'] \\
 & & {} &
\end{tikzcd}$$

The $\mathcal{U}_q(\hat{sl}_4)$-crystal $\mathcal{M}(e^{2\varpi_1}Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.



The $\mathcal{U}_q(\hat{sl}_4)$-crystal $\mathcal{M}(e^{2\varpi_1+\delta}Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1})$.

Appendix B

Action on the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$

The Appendix is devoted to the proof of Proposition 3.2.15. We study the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ ($a \in \mathbb{C}^*$) and we determine the action of the quantum toroidal algebra on it. It proves Conjecture 3.2.13 for the particular case $s = 1$ and $\ell = 1, n$ (see Proposition 3.2.16).

Let $V(e^{\Lambda_1}Y_{1,a})$ (resp. $V(e^{-\Lambda_0}Y_{0,aq}^{-1})$) be the simple ℓ-highest weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module of ℓ-highest weight $e^{\Lambda_1}Y_{1,a}$ (resp. the simple ℓ-lowest weight $\mathcal{U}_q(sl_{n+1}^{tor})$-module of ℓ-lowest weight $e^{-\Lambda_0}Y_{0,aq}^{-1}$). Recall that $T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ is defined by

$$T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1}) = \mathcal{U}_q(sl_{n+1}^{tor}) \cdot \left(v_{Y_{1,a}} \otimes w_{Y_{0,aq}^{-1}}\right) \subseteq V(e^{\Lambda_1}Y_{1,a}) \otimes V(e^{-\Lambda_0}Y_{0,aq}^{-1})$$

where $v_{Y_{1,a}}$ (resp. $w_{Y_{0,aq}^{-1}}$) is the ℓ-highest weight vector of $V(e^{\Lambda_1}Y_{1,a})$ (resp. the ℓ-lowest weight vector of $V(e^{-\Lambda_0}Y_{0,aq}^{-1})$).

To simplify the notations, one can assume in our computations that $a = 1$. The other representations $T(e^{\varpi_1}Y_{1,a}Y_{0,aq}^{-1})$ can be obtained from $T(e^{\varpi_1}Y_{1,1}Y_{0,q}^{-1})$ by twisting the action by t_a. We will denote variables Y_{i,q^l} only by $Y_{i,l}$ in the following ($i \in I$, $l \in \mathbb{Z}$). We still use the modulo $(n+1)$ convention: for all $i \in \mathbb{Z}$, we set $x_i^{\pm}(z) = x_{\bar{i}}^{\pm}(z)$ and $Y_{i,*} = Y_{\bar{i},*}$.

To study the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $T(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})$, we determine the action in each $\mathcal{U}_q(\hat{sl}_2)'$-direction on the met vectors. Let us justify some computations done in the following. We will have to study $\mathcal{U}_q(\hat{sl}_2)'$-modules generated by ℓ-highest weight vectors of ℓ-highest weight of the form Y_l or Y_lY_m ($l, m \in \mathbb{Z}, l \neq m$). By the theory of Weyl modules [CP01], these representations are known: they are isomorphic to the simple ℓ-highest weight $\mathcal{U}_q(\hat{sl}_2)'$-modules of ℓ-highest weights Y_l and Y_lY_m respectively, which are thin. We will use this fact in our computations. In particular we will see that the $\mathcal{U}_q(sl_{n+1}^{tor})$-module $T(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})$ is also thin. We will determine a basis of ℓ-weight vectors of $T(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1})$, indexed by its set of ℓ-weights $\mathcal{M}(T(e^{\varpi_1}Y_{1,0}Y_{0,1}^{-1}))$.

Let us consider the action on the vector $v_{Y_{1,0}} \otimes w_{Y_{0,1}^{-1}}$: we have

$$\begin{aligned}
x_0^+(z) \cdot \left(v_{Y_{1,0}} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(z) \cdot v_{Y_{1,0}} \otimes w_{Y_{1,0}^{-1}Y_{3,0}^{-1}Y_{0,-1}}, \\
x_1^-(z) \cdot \left(v_{Y_{1,0}} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(qz) \cdot v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{0,1}^{-1}}, \\
x_i^\pm(z) \cdot \left(v_{Y_{1,0}} \otimes w_{Y_{0,1}^{-1}}\right) &= 0 \text{ in all the other cases,}
\end{aligned}$$

where we set $v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} = x_{1,0}^- \cdot v_{Y_{1,0}}$ and $w_{Y_{1,0}^{-1}Y_{3,0}^{-1}Y_{0,-1}} = x_{0,0}^+ \cdot w_{Y_{0,1}^{-1}}$.

We have to determine the action on $v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{0,1}^{-1}}$: it is given by

$$\begin{aligned}
x_0^+(z) \cdot \left(v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{0,1}^{-1}}\right) &= \psi(q^2z)\delta(z) \cdot v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{1,0}^{-1}Y_{3,0}^{-1}Y_{0,-1}} \\
&= \psi(q^2)\delta(z) \cdot v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{1,0}^{-1}Y_{3,0}^{-1}Y_{0,-1}} = 0, \\
x_0^-(z) \cdot \left(v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^2z)\psi(q^2z)^{-1} \cdot v_{Y_{0,3}^{-1}Y_{2,1}Y_{3,2}} \otimes w_{Y_{0,1}^{-1}} \\
&= \delta(q^2z)\psi(1)^{-1} \cdot v_{Y_{0,3}^{-1}Y_{2,1}Y_{3,2}} \otimes w_{Y_{0,1}^{-1}} = 0, \\
x_1^+(z) \cdot \left(v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(qz) \cdot v_{Y_{1,0}} \otimes w_{Y_{0,1}^{-1}}, \\
x_2^-(z) \cdot \left(v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^2z) \cdot v_{Y_{0,1}Y_{2,3}^{-1}Y_{3,2}} \otimes w_{Y_{0,1}^{-1}}, \\
x_i^\pm(z) \cdot \left(v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}} \otimes w_{Y_{0,1}^{-1}}\right) &= 0 \text{ in all the other cases,}
\end{aligned}$$

where we set $v_{Y_{0,3}^{-1}Y_{2,1}Y_{3,2}} = x_{0,0}^- \cdot v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}}$ and $v_{Y_{0,1}Y_{2,3}^{-1}Y_{3,2}} = x_{2,0}^- \cdot v_{Y_{0,1}Y_{1,2}^{-1}Y_{2,1}}$.

These formulas generalize by induction: we show that for all $j \geq 1$, we have

$$\begin{aligned}
x_j^+(z) \cdot \left(v_{Y_{0,1}Y_{j,j+1}^{-1}Y_{j+1,j}} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^jz)v_{Y_{0,1}Y_{j-1,j}^{-1}Y_{j,j-1}} \otimes w_{Y_{0,1}^{-1}}, \\
x_{j+1}^-(z) \cdot \left(v_{Y_{0,1}Y_{j,j+1}^{-1}Y_{j+1,j}} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^{j+1}z)v_{Y_{0,1}Y_{j+1,j+2}^{-1}Y_{j+2,j+1}} \otimes w_{Y_{0,1}^{-1}}, \\
x_j^\pm(z) \cdot \left(v_{Y_{0,1}Y_{j,j+1}^{-1}Y_{j+1,j}} \otimes w_{Y_{0,1}^{-1}}\right) &= 0 \text{ in all the other cases,}
\end{aligned} \tag{B.1}$$

where we set $v_{Y_{0,1}Y_{i,i+1}^{-1}Y_{i+1,i}} = x_{i+1,0}^- \cdot v_{Y_{0,1}Y_{i,i+1}^{-1}Y_{i+1,i}}$. To show (B.1) similar computations as above can be done, except for vectors of the form

$$v_{Y_{0,1}Y_{n,k(n+1)}^{-1}Y_{0,k(n+1)-1}} \text{ and } v_{Y_{0,1}Y_{0,k(n+1)+1}^{-1}Y_{1,k(n+1)}} \; (k \in \mathbb{N}^*)$$

for which it is more complicated. Let us explain more precisely what it is happening for them.

Let us begin by the particular case $n = 3$ and $k = 1$. Then $\hat{\mathcal{U}}_0 \cdot v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}$ is isomorphic to the Kirillov-Reshetikhin $\mathcal{U}_q(\hat{sl}_2)'$-module of ℓ-highest weight $Y_{0,3}Y_{0,1}$. Hence we have

$$\begin{aligned}
x_0^+(z)\cdot\left(v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{0,1}^{-1}}\right) &= \psi(q^2z)\psi(q^4z)\delta(z)\cdot v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{1,0}^{-1}Y_{3,0}^{-1}Y_{0,-1}}\\
&= \psi(q^2)\psi(q^4)\delta(z)\cdot v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{1,0}^{-1}Y_{3,0}^{-1}Y_{0,-1}} = 0,\\
x_0^-(z)\cdot\left(v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^4z)\psi(q^2z)^{-1}\cdot v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}\otimes w_{Y_{0,1}^{-1}}\\
&= \delta(q^4z)\psi(q^{-2})^{-1}\cdot v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}\otimes w_{Y_{0,1}^{-1}},\\
x_3^+(z)\cdot\left(v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^3z)\cdot v_{Y_{0,1}Y_{2,3}^{-1}Y_{3,2}}\otimes w_{Y_{0,1}^{-1}},\\
x_i^\pm(z)\cdot\left(v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{0,1}^{-1}}\right) &= 0 \text{ in all the other cases,}
\end{aligned}$$

where we set $v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}} = x_{0,0}^-\cdot v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}$.

We determine the action on the vectors $v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}\otimes w_{Y_{0,1}^{-1}}$:

$$\begin{aligned}
x_0^+(z)\cdot\left(v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}\otimes w_{Y_{0,1}^{-1}}\right) &= \psi(q^{-2})\delta(q^4z)\cdot v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{0,1}^{-1}}\\
&\quad +\psi(q^2)\psi(q^6)^{-1}\delta(z)\cdot v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{1,0}^{-1}Y_{3,0}^{-1}Y_{0,-1}}\\
&= \psi(q^{-2})\delta(q^4z)\cdot v_{Y_{0,1}Y_{3,4}^{-1}Y_{0,3}}\otimes w_{Y_{0,1}^{-1}},\\
x_0^-(z)\cdot\left(v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}\otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^2z)\psi(q^2z)^{-1}\cdot v_{Y_{0,3}^{-1}Y_{0,5}^{-1}Y_{1,4}Y_{1,2}Y_{3,2}}\otimes w_{Y_{0,1}^{-1}} = 0,\\
x_1^-(z)\cdot\left(v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}\otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^5z)\cdot v_{Y_{0,1}Y_{1,6}^{-1}Y_{2,5}}\otimes w_{Y_{0,1}^{-1}},\\
x_i^\pm(z)\cdot\left(v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}\otimes w_{Y_{0,1}^{-1}}\right) &= 0 \text{ in all the other cases,}
\end{aligned}$$

where we set $v_{Y_{0,3}^{-1}Y_{0,5}^{-1}Y_{1,4}Y_{1,2}Y_{3,2}} = x_{0,0}^-\cdot v_{Y_{0,1}Y_{0,5}^{-1}Y_{1,4}}$.

From now we assume that $n\neq 3$ or $k\neq 1$. Then $\hat{\mathcal{U}}_0\cdot v_{Y_{0,1}Y_{n,k(n+1)}^{-1}Y_{0,k(n+1)-1}}$ is isomorphic to the simple ℓ-highest weight $\mathcal{U}_q(\hat{sl}_2)'$-module of ℓ-highest weight $Y_{0,k(n+1)-1}Y_{0,1}$. This module is studied in [Her10, Man12c]. We recall these results in the following example.

Example B.0.1. *Let $a,b\in\mathbb{Z}$ be such that $a\neq b$ and $a\neq b\pm 2$. Consider the simple ℓ-highest weight $\mathcal{U}_q(\hat{sl}_2)'$-module V of ℓ-highest weight $Y_{1,a}Y_{1,b}$. We have*

$$\chi_q(V) = Y_{1,a}Y_{1,b} + Y_{1,a}Y_{1,b+2}^{-1} + Y_{1,a+2}^{-1}Y_{1,b} + Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}.$$

In particular, there exists a basis

$$\{v_{Y_{1,a}Y_{1,b}}, v_{Y_{1,a+2}^{-1}Y_{1,b}}, v_{Y_{1,a}Y_{1,b+2}^{-1}}, v_{Y_{1,a+2}^{-1}Y_{1,b+2}^{-1}}\}$$

where the action is given by

$$
\begin{aligned}
x_1^+(z)\cdot v_{Y_{1,a}Y_{1,b}} &= 0,\\
x_1^-(z)\cdot v_{Y_{1,a}Y_{1,b}} &= \psi(q^{b-a})\delta(q^{a+1}z)\cdot v_{Y^{-1}_{1,a+2}Y_{1,b}} + \psi(q^{a-b})\delta(q^{b+1}z)\cdot v_{Y_{1,a}Y^{-1}_{1,b+2}},\\
x_1^+(z)\cdot v_{Y^{-1}_{1,a+2}Y_{1,b}} &= \delta(q^{a+1}z)\cdot v_{Y_{1,a}Y_{1,b}},\\
x_1^-(z)\cdot v_{Y^{-1}_{1,a+2}Y_{1,b}} &= \delta(q^{b+1}z)\cdot v_{Y^{-1}_{1,a+2}Y^{-1}_{1,b+2}},\\
x_1^+(z)\cdot v_{Y_{1,a}Y^{-1}_{1,b+2}} &= \delta(q^{b+1}z)\cdot v_{Y_{1,a}Y_{1,b}},\\
x_1^-(z)\cdot v_{Y_{1,a}Y^{-1}_{1,b+2}} &= \delta(q^{a+1}z)\cdot v_{Y^{-1}_{1,a+2}Y^{-1}_{1,b+2}},\\
x_1^+(z)\cdot v_{Y^{-1}_{1,a+2}Y^{-1}_{1,b+2}} &= \psi(q^{b-a})\delta(q^{b+1}z)\cdot v_{Y^{-1}_{1,a+2}Y_{1,b}} + \psi(q^{a-b})\delta(q^{a+1}z)\cdot v_{Y_{1,a}Y^{-1}_{1,b+2}},\\
x_1^-(z)\cdot v_{Y^{-1}_{1,a+2}Y^{-1}_{1,b+2}} &= 0,
\end{aligned}
$$

and with v_m of ℓ-weight m for $m = Y_{1,a}Y_{1,b},\dots,Y^{-1}_{1,a+2}Y^{-1}_{1,b+2}$.

So let us denote

$$
\begin{aligned}
m &= Y_{0,1}Y^{-1}_{n,k(n+1)}Y_{0,k(n+1)-1},\\
m_1 &= Y^{-1}_{0,3}Y_{1,2}Y_{n,2}Y^{-1}_{n,k(n+1)}Y_{0,k(n+1)-1},\\
m_2 &= Y_{0,1}Y^{-1}_{0,k(n+1)+1}Y_{1,k(n+1)}
\end{aligned}
$$

and let v_{m_1} and v_{m_2} be such that

$$
\begin{aligned}
x_0^-(z)\cdot v_m &= \psi(q^{k(n+1)-2})\delta(q^2z)\psi(q^2z)^{-1}\cdot v_{m_1}\\
&\quad +\psi(q^{2-k(n+1)})\delta(q^{k(n+1)}z)\psi(q^2z)^{-1}\cdot v_{m_2}.
\end{aligned}
$$

The action on $v_m\otimes w_{Y^{-1}_{0,1}}$ is:

$$
\begin{aligned}
x_0^+(z)\cdot\left(v_m\otimes w_{Y^{-1}_{0,1}}\right) &= \psi(q^2z)\psi(q^{k(n+1)}z)\delta(z)\cdot v_m\otimes w_{Y^{-1}_{1,0}Y^{-1}_{n,0}Y_{0,-1}} = 0,\\
x_0^-(z)\cdot\left(v_m\otimes w_{Y^{-1}_{0,1}}\right) &= \psi(q^{k(n+1)-2})\delta(q^2z)\psi(q^2z)^{-1}\cdot v_{m_1}\otimes w_{Y^{-1}_{0,1}}\\
&\quad +\psi(q^{2-k(n+1)})\delta(q^{k(n+1)}z)\psi(q^2z)^{-1}\cdot v_{m_2}\otimes w_{Y^{-1}_{0,1}}\\
&= \delta(q^{k(n+1)}z)\cdot v_{m_2}\otimes w_{Y^{-1}_{0,1}},\\
x_n^+(z)\cdot\left(v_m\otimes w_{Y^{-1}_{0,1}}\right) &= \delta(q^{k(n+1)-1}z)\cdot v_{Y_{0,1}Y^{-1}_{n-1,k(n+1)-1}Y_{n,k(n+1)-2}}\otimes w_{Y^{-1}_{0,1}},\\
x_i^\pm(z)\cdot\left(v_m\otimes w_{Y^{-1}_{0,1}}\right) &= 0 \text{ in all the other cases.}
\end{aligned}
$$

The action on $v_{m_2} \otimes w_{Y_{0,1}^{-1}}$ is:

$$\begin{aligned}
x_0^+(z) \cdot \left(v_{m_2} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^{k(n+1)} \cdot v_m \otimes w_{Y_{0,1}^{-1}} \\
&\quad + \psi(q^2 z)\psi(q^{k(n+1)+2} z)^{-1} \delta(z) \cdot v_{m_2} \otimes w_{Y_{1,0}^{-1} Y_{n,0}^{-1} Y_{0,-1}} \\
&= \delta(q^{k(n+1)}) \cdot v_m \otimes w_{Y_{0,1}^{-1}}, \\
x_0^-(z) \cdot \left(v_{m_2} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^2 z)\psi(q^2 z)^{-1} \cdot v_{Y_{0,3}^{-1} Y_{0,k(n+1)+1}^{-1} Y_{1,k(n+1)} Y_{1,2} Y_{n,2}} \otimes w_{Y_{0,1}^{-1}} = 0, \\
x_1^-(z) \cdot \left(v_{m_2} \otimes w_{Y_{0,1}^{-1}}\right) &= \delta(q^{k(n+1)+1}) \cdot v_{Y_{0,1} Y_{1,k(n+1)+2}^{-1} Y_{2,k(n+1)+1}} \otimes w_{Y_{0,1}^{-1}}, \\
x_i^\pm(z) \cdot \left(v_{m_2} \otimes w_{Y_{0,1}^{-1}}\right) &= 0 \text{ in all the other cases,}
\end{aligned}$$

where we set $v_{Y_{0,3}^{-1} Y_{0,k(n+1)+1}^{-1} Y_{1,k(n+1)} Y_{1,2} Y_{n,2}} = x_{0,0}^- \cdot v_{m_2}$.

Hence we have shown formulas (B.1) for the action of the negative part of $\mathcal{U}_q(sl_{n+1}^{tor})$. The same computations hold for the action of $\mathcal{U}_q(sl_{n+1}^{tor})^+$. We determine in this way a basis of $T(e^{\varpi_1} Y_{1,0} Y_{0,1})$ formed by ℓ-weight vectors. The Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ acts on it by diagonalizable operators with a simple joint spectrum. Then $T(e^{\varpi_1} Y_{1,0} Y_{0,1}))$ is a thin $\mathcal{U}_q(sl_{n+1}^{tor})$-module. We get Proposition 3.2.15 by twisting the action on $T(e^{\varpi_1} Y_{1,0} Y_{0,1})$ by the automorphism t_a ($a \in \mathbb{C}^*$).

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